BY THE SAME AUTHOR.

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AN

ELEMENTARY TREATISE

ON

GEOMETRICAL OPTICS.

BY

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PREFACE.

The main object of the present treatise is to supply a text-book on Geometrical Optics to students reading for the Mathematical Tripos at Cambridge, who do not wish to proceed much beyond those portions of the subject which are required for the first part of the Tripos Examination.

The investigations are therefore not carried beyond *first approximations*. The discussion of the position of the foci of obliquely incident pencils has, however, been brought within this boundary, instead of being derived from the second approximations for direct pencils.

The Author hopes that the book may be useful to a wider class of students, not residing in any University, by giving to them a concise view of the

mathematical explanation of instruments, with the practical details of which they are familiar.

The Author wishes to express his acknowledgements to several friends, for hints and suggestions, and especially to Mr W. M. Spence, Fellow of Pembroke College, Cambridge, for his valuable assistance in revising the book as it went through the press.

College of Physical Science,
Newcastle-upon-Tyne,
September, 1872.

CONTENTS.

CHAPTER I.

LAWS OF REFLECTION AND REFRACTION.

1. THE subject of Optics divides itself naturally into two distinct parts.

One of these consists in the deduction by geometrical or analytical methods of the consequences of a few well ascertained laws which govern the simplest phenomena of light. The second consists in the explanation of the mechanical or physical causes which produce those phenomena. These two branches of the subject are usually known as Geometrical and Physical Optics respectively, and it is with the former exclusively that the present treatise is concerned. We shall not discuss the physical causes of the propagation of light, but taking certain laws for granted, we shall endeavour to trace out some of their more interesting and useful consequences. It will be necessary to commence with a few important definitions and explanations.

2. When we are in a place exposed either to the light of the sun or any artificial source of light, we are sensible of the existence of objects surrounding us. If the light of the sun be excluded or the artificial light extinguished, we cease to be able to perceive by sight anything that is near to us. Such bodies as the sun or a lighted lamp have therefore the property of rendering us sensible by sight not only of their own existence, but of that of all other bodies on which they shed what we call their light. Bodies which have this power are called self-luminous bodies. On

the other hand, bodies which require the presence of some self-luminous body in order to render us aware of their existence by sight, are called non-luminous or dark bodies.

3. We assume that the sensation of sight is produced by something (not necessarily material) which comes from the thing seen and enters the eye. Experiment shows that it proceeds in straight lines. We assume farther that this something, which we shall in future call light, proceeds to the eye from every material point of any body which is seen. The quantity of light which proceeds from any material point of a body to the eye we shall call a pencil of light. We shall also suppose that the form of this pencil is a cone, whose vertex is the luminous point, and whose base is the portion of the eye which admits light.

4. If we suppose the vertical angle of this cone to be indefinitely diminished, we get a certain quantity of light which may be considered as a straight line, and is called *a ray*. It is not necessary for our purposes that such a small quantity of light shall be actually able to exist separately, but it is evident that we may suppose the pencils we have before considered to consist of an indefinite number of small portions, such as we have defined as rays.

We may then give the following definitions:

(1) A pencil of light is the portion of light, by means of which a given material point of any object might be seen by an eye suitably placed. It is generally considered to be of a conical form with the material point at its vertex.

If the material point be at an indefinitely great distance, the cone will assume a cylindrical form.

(2) A ray of light is the limiting form of a pencil of light when the solid angle at the vertex of the cone is indefinitely diminished. It is usually considered to be a line; and in accordance with a remark previously made, it is a *straight* line as long as it continues in the same medium. A pencil is conceived to be made up of an infinite number of such rays.

5. We know by experience that if there be nothing but air or vacuum between us and any luminous object, the

light of that object is able to reach our eyes. If we interpose a piece of glass, or ice, or a rectangular vessel of glass containing clear water between our eyes and the luminous object, the light is still able to produce the sense of sight.

If, on the other hand, we hold up a piece of wood or iron between our eyes and the object, the latter becomes invisible to us, the light not being able to traverse the wood or iron.

We thus get an optical distinction between different classes of bodies. Some bodies permit light to traverse them more or less freely and regularly: others refuse to allow it to pass at all. Bodies of the former class are called transparent bodies; of the latter, opaque.

6. There are some bodies, as alabaster, porcelain, which, when held up between our eyes and a strongly luminous body, as the sun, allow light to pass in an irregular way, but do not permit us to see the luminous body distinctly. Such bodies are called translucent. Light transmitted through such bodies will not be farther considered in this book, as it obeys no simple geometrical laws.

7. When a body is considered with reference to its power of transmitting light, it is usually called a medium.

8. When a pencil of light proceeding in one medium is incident on the surface of another transparent medium, it is usually divided into three parts.

(1) A portion is *reflected* back into the original medium according to a law to be hereafter stated.

(2) A portion passes into the new medium according to another law to be hereafter stated, and is said *to be refracted* into the new medium.

(3) A third portion is employed in rendering visible the surface which separates the two media.

For instance, when the sun is shining on a window, the sun's light comes through the air, and is incident on the plane surface of the glass. Some of this light goes into the glass, and again passes out into the air on the other side,

as is proved by the luminous patch resembling in its general shape the window, which is seen within the room; and also by the fact that an observer within the room can see the sun distinctly. This is the second or refracted portion.

Some of the light is reflected externally, as is shown by the fact that an observer outside the room can see the sun's image reflected in the window just as in a looking glass, only not so vividly.

This is the first or reflected portion.

A third portion is employed in rendering the window visible, and is said to be scattered. The observer outside will be able to see the specks and marks on the surface of the glass by means of this portion. If the glass of the window were perfectly smooth and clean, this portion would probably not exist, and the whole of the light would be either reflected or refracted.

We shall at present only consider the reflected and refracted portions.

9. Before stating the laws which regulate the directions of the reflected and refracted portions corresponding to a given incident pencil, we must define a few terms which will be of constant occurrence.

The straight line drawn at right angles to a plane at a given point is called the *normal* to the plane at that point.

We know, from the example of the earth, which is really spherical, but of which any small portion appears to be a plane, that any portion of a spherical surface sufficiently small in comparison with the size of the whole sphere may be considered as a plane. The normal to a spherical or other curved surface at any point is the line drawn through that point at right angles to the plane, with which the small part of the surface immediately surrounding this point may be supposed to coincide. In the case of a sphere the reader must assume, if it be not obvious to him, that the normal at any point is the straight line joining that point with the centre of the sphere.

10. If a ray of light in its progress from the original point from which it emanates comes to the surface of a dif-

ferent medium from that in which it is at first propagated, it is said *to be incident* on the second medium.

In this case the plane containing the incident ray and the normal to the surface which separates the second medium from the first is called *the plane of incidence* of the ray.

The angle between the incident ray and the normal to the bounding surface is called *the angle of incidence* of the ray.

11. When a ray is so incident, the following laws govern the directions of the two portions of it which are respectively reflected and refracted.

(1) The reflected and refracted rays both lie in the plane of incidence of the original ray and on the opposite side of the normal to the surface to that on which the incident ray lies.

(2) The angle which the reflected ray makes with this normal is equal to the angle which the incident ray makes with the normal. If we agree to call the angle between the reflected ray and the normal the angle of reflection, this law may be concisely stated thus—the angles of incidence and reflection are equal.

(3) The sine of the angle which the incident ray makes with the normal bears a constant ratio to the sine of the angle which the refracted ray makes with the normal to the surface; constant, that is, for the same kind of light and the same media.

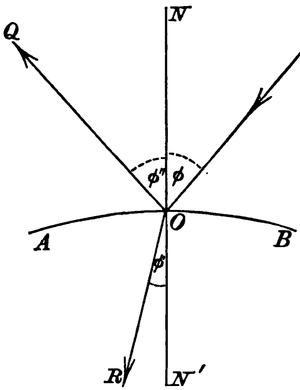
This constant ratio is called the *index of refraction* or *refractive index* from the first medium into the second.

If the first medium be a vacuum, this constant ratio is called *the absolute index of refraction* of the second medium.

If the second medium be denser in substance than the first, this ratio or refractive index is greater than unity; as, for instance, when light passes from air to glass.

12. Thus, for instance, let us suppose the plane of incidence of a ray PO to coincide with the plane of the

paper, and let NON' be the normal to the surface dividing the media, at the point O where the ray is incident. Let AOB be the line of intersection of this bounding surface with the plane of the paper.



By the first law, the reflected ray OQ and the refracted ray OR will both lie in the plane of the paper, and to the left of ON, as we have drawn OP to the right of ON. Also OQ lies above AOB and OR lies below AOB.

The angle PON is the angle of incidence.

The angle NOQ is the angle of reflection.

The angle $N'OR$ is the angle of refraction.

If we call these angles ϕ, ϕ'', ϕ' respectively we have by the second law

$$\phi = \phi'',$$

and by the third law

$$\frac{\sin\phi}{\sin\phi'} = \text{a constant which we may call } \mu.$$

These equations determine ϕ'' and ϕ' when ϕ is known.

If the second medium be of a denser nature than the first, μ is greater than unity, as has been already remarked;

ϕ is consequently greater than ϕ'. Thus, in passing into a denser medium a ray of light is bent towards the normal to the bounding surface; on the other hand, in passing from a denser medium into one less dense, the light is bent away from the normal.

13. It is usual to consider separately the two portions into which the ray is divided, and to speak of a ray as reflected, or refracted, at a given surface when we mean that we are only going to discuss the position and direction of the reflected or refracted portions respectively.

14. It is found experimentally that if a ray PO when incident on the surface of a second medium be refracted in the direction OR, a ray coming in the second medium in the direction RO will be refracted along the line OP.

If we call the two media A and B respectively, and denote by ${}_A\mu_B$ the index of refraction from the first medium into the second, and by ${}_B\mu_A$ the index of refraction from the second medium into the first, we have with our previous notation

$$\frac{\sin\phi}{\sin\phi'} = {}_A\mu_B,$$

$$\frac{\sin\phi'}{\sin\phi} = {}_B\mu_A,$$

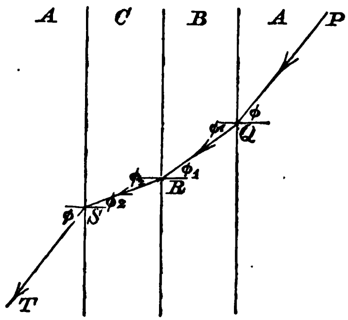
whence

$${}_B\mu_A = \frac{1}{{}_A\mu_B} \qquad \text{(1)}.$$

Again, it is found by experiment that a ray of light after passing through any number of media bounded by parallel planes, as for instance through a number of plates of glass of different kinds, when it comes again into a medium of the same nature as that from which it originally was incident on the plates, will be in a direction parallel to its original one.

Thus let A be the original medium, and let $PQRST$ be the course of a ray in passing through portions of two media B and C bounded by parallel planes.

Let ϕ be the angle of incidence on B, ϕ_1 the angle of refraction into B; it is evident that ϕ_1 will also be the



angle of incidence on C. Let ϕ_2 be the angle of refraction into C, which will also be the angle of incidence on A, while by the above remarks ϕ will be the angle of refraction into A again. Then we have, with the notation explained above,

$$\frac{\sin\phi}{\sin\phi_1} = {}_A\mu_B, \quad \frac{\sin\phi_1}{\sin\phi_2} = {}_B\mu_C, \quad \frac{\sin\phi_2}{\sin\phi} = {}_C\mu_A,$$

$$\therefore \; {}_A\mu_B \times {}_B\mu_C \times {}_C\mu_A = 1,$$

which, since by means of equation (1)

$${}_A\mu_C = \frac{1}{{}_C\mu_A},$$

gives us

$${}_A\mu_C = {}_A\mu_B \times {}_B\mu_C \; \text{.....................} \; (2).$$

One great use of this equation is to connect the relative index of refraction between two media with their absolute refractive indices.

Thus let ${}_V\mu_A$ be the index of refraction from vacuum into A, that is the absolute refractive index of A (Art. 11), and let ${}_V\mu_B$ be similarly the absolute refractive index of B. Then by the above equation

$${}_V\mu_A = {}_V\mu_B \times {}_B\mu_A,$$

$$\left.\begin{array}{ll} \therefore & {}_B\mu_A = \dfrac{{}_V\mu_A}{{}_V\mu_B} \\ \text{Similarly, or by (1),} & \\ & {}_A\mu_B = \dfrac{{}_V\mu_B}{{}_V\mu_A} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

These results (1), (2), (3) are of considerable importance.

15. Let us again consider the equation

$$\sin\phi = \mu \sin\phi',$$

or

$$\sin\phi' = \frac{1}{\mu}\sin\phi,$$

which connects the angles which the directions of the ray in the first and second medium respectively make with the normal to the separating surface. Assume also that μ is greater than unity, so that the second medium is denser than the first.

It is evident that, if ϕ be given, since $\sin\phi'$ is less than $\sin\phi$, and therefore less than unity, a real value of ϕ' can always be found. Hence if a ray of light be incident from a rarer medium on a denser, the ordinary law of refraction always gives a direction for the refracted ray.

If, on the other hand, ϕ' be given, $\sin\phi$ may happen to be greater than unity and no real value of ϕ can be found. This will be the case if

$$\sin\phi' > \frac{1}{\mu},$$

and for values of ϕ' exceeding the value given by the equation

$$\sin\phi' = \frac{1}{\mu} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

no direction for the refracted ray is given by the ordinary law.

In the case of any two media, the greatest angle at which a ray, proceeding in the denser medium, can be incident on the rarer so as to be refracted into the rarer, is called *the critical angle between those media.* Its value

is given by the equation (1), where μ indicates the refractive index from the rarer to the denser medium.

If the rarer medium be a *vacuum*, μ will be the absolute refractive index of the denser medium and the corresponding critical angle is called the *absolute critical angle*, or sometimes simply *the critical angle* of the denser medium.

16. It is found by experiment that when a ray of light is incident on a medium rarer than that in which it is moving, at an angle greater than the critical angle between those media, the whole of the light is reflected; the refracted portion does not exist. This is known as the phenomenon of *total internal reflection.*

17. We have hitherto considered the second medium to be a transparent medium capable of transmitting light. If it be opaque, as when light is incident on the polished surface of metal, the refracted portion does not exist, or, at any rate, does not make us sensible of its existence. There is however in this case a reflected ray following the laws given previously in Art. 11. The amount of light in the reflected ray is not so great as in the incident ray, and is much less than in the case of total internal reflection given in the last Article.

We have thus far considered the modification produced in a single ray by reflection or refraction at a surface. We have to consider in the next Chapter the more complicated modifications produced in a pencil by such refractions or reflections.

EXAMPLES. CHAPTER I.

1. Find the angle of refraction, when a ray is refracted from vacuum into a medium whose refractive index is $\sqrt{2}$, the angle of incidence being 45^0.

2. The angle of incidence being 60^0, and the index of refraction being $\sqrt{3}$; find the angle of refraction.

3. The absolute refractive indices of two media being $\sqrt{(5)}-1$ and 2 respectively, find the angle of refraction of a ray incident in the first medium on the second; the angle of incidence being 30^0.

4. A person looking over the rim of a cylindrical cup is unable to see the bottom. When water is poured in, part of the base of the cup becomes visible. Explain this.

5. The height of a cylindrical cup is 4 inches, the diameter of its base is 3 inches. A person looks over its rim so that the lowest point of the opposite side which he can see is $2\frac{1}{4}$ inches below the top. The cup is filled with water; looking in the same direction he can just see the point of the base farthest from him. Find the refractive index of water.

6. A ray of light is incident on a refracting surface whose refractive index is μ, at an angle $\tan^{-1}\mu$. Show that the angle of refraction is $\tan^{-1}\frac{1}{\mu}$.

7. A ray of light is incident on a refracting sphere, whose refractive index is $\sqrt{3}$. It is refracted into the sphere, and when it is incident on the inner surface of the sphere, part is reflected internally, and part is refracted out into vacuum. Show that if the original angle of incidence be 60^0, these two parts are at right angles to each other.

8. In the last question, show that if the part internally reflected be again incident internally and be refracted out into vacuum, its final course will be parallel to that of the ray first incident.

9. A ray is incident on a refracting sphere whose refractive index is $\frac{3}{2}$, at an angle whose sine is $\frac{3\sqrt{3}}{4\sqrt{2}}$. Show that if the ray be refracted into the sphere, that portion of it which emerges after having been twice internally reflected will be in the same direction as the original ray.

10. Show that a pencil of light emanating from the focus of a prolate spheroid whose inner surface is reflecting, will be accurately reflected to a point. Show also that a pencil of light emanating from the focus of a paraboloid of revolution whose concave surface is reflecting, will be reflected as a pencil of parallel rays.

11. A luminous point is placed at the focus of an ellipse, the inner side of which can reflect light. Prove that any ray after two reflections from the curve will return to the focus from which it started, and will have travelled over a distance equal to twice the axis major of the ellipse.

12. A ray passes through four media whose absolute refractive indices are $\sqrt{3}$, $\sqrt{6}$, $\sqrt{2}$, and $\dfrac{\sqrt{15}+\sqrt{3}}{\sqrt{2}}$ respectively, the angle of incidence on the second medium from the first being 45^0. Find the angles of refraction into the second, third, and fourth media respectively.

13. A ray proceeding from a point P, and incident on a plane surface at O, is partly reflected to Q, and partly refracted to R: if the angles POQ, POR, QOR be in Arithmetic Progression, find the angle of incidence, μ being the index of refraction. Explain the result when $\mu = 2$.

14. If the angles POQ, QOR and ROP be in Arithmetic Progression, in the last question, find the angle of incidence.

15. ABC is a triangle, the interior of the sides of which can reflect light. In the side BC are two small holes at P and Q. Find the position of a point outside the triangle, such that a ray of light proceeding from it so as to enter through P may be reflected so as to pass out through Q, and also a ray from it entering through Q may be reflected out through P.

16. A ray of light is incident on a concave refracting spherical surface of radius r. Its direction before refraction cuts the axis of the surface at a distance μr from the centre of the sphere. Show that after refraction its direction will cut the axis at a distance $\dfrac{r}{\mu}$ from the centre of the sphere.

17. A ray of light proceeds from one point P of an ellipse, and falls upon a reflecting plane at one focus. Find the position of the plane, that after reflection the ray may pass through a given point Q of the ellipse.

18. If a ray proceeding from the extremity of one diameter of an ellipse be reflected at the curve so as to pass through the other extremity of this diameter; prove that the length of the path of the ray is the same whatever diameter be taken.

19. In the last question, prove that, if ϕ, ϕ' be the eccentric angles of the extremity of the diameter and the point where the reflection takes place, $\tan\phi \, . \tan\phi' = -\frac{b^2}{a^2}$; a and b being the semiaxes of the ellipse.

20. A ray of light emanating from a point P of an ellipse, after one reflection at the inner side of the curve, is again reflected at the opposite extremity of the diameter through P. Prove that after another reflection it will return to P, and then retrace its path, having described a parallelogram.

21. A ray of light is incident upon a refracting sphere, whose refractive index is $\sqrt{3}$. The refracted ray and the incident ray produced cut the sphere in points the arc joining which subtends an angle of 60^0 at the centre. Find the angle of incidence.

22. The length of the path of a ray which passes through two plates of different media in contact, bounded by parallel planes, is $a\sqrt{3}$ in the first medium and $2c$ in the second. The thicknesses of the plates being a and c respectively, find the refractive index from the first plate into the second.

23. Prove that light which has been refracted into a sphere from vacuum can never be totally internally reflected.

24. If light be incident on the curved surface of a hemisphere of a refracting medium in a direction parallel to its axis, shew that there will be no total internal reflection at the plane surface, unless the refractive index is greater than $\sqrt{2}$.

25. Three plane mirrors are placed so as to be all perpendicular to the same plane, their intersections with which form an acute-angled triangle; a ray proceeding from a certain point in this plane after one reflection at each of the mirrors proceeds on its original course. Show that the point must lie on the perimeter of the triangle formed by joining the feet of the perpendiculars from the angular points of the original triangle on the opposite sides.

26. The concave side of an equiangular spiral being polished, prove that a ray of light once a tangent to the spiral will be always a tangent to the spiral, however often it may be reflected at the curve.

27. A ray of light is incident on a portion of a refracting medium in the shape of a prolate spheroid; the eccentricity of

the generating ellipse is e, and the refractive index is $\frac{1}{e}$. The ray being incident parallel to the axis of the spheroid, show that after refraction it will pass through one focus.

28. A pencil of rays emanates from a point at a distance μr from the centre of a refracting sphere whose radius is r and refractive index μ. Prove that the extreme incident rays on emergence intersect a screen touching the sphere at the point opposite to the origin of light, in a circle, whose radius is

$$\frac{2-\mu}{2+\mu}\sqrt{\left(\frac{\mu+1}{\mu-1}\right)}\,.\,r.$$

29. If a ray of light be reflected at a plane surface, the incident and reflected rays make equal angles with any line lying in that plane.

30. If a ray of light be refracted at a plane surface, the cosines of the angles which the incident and refracted rays respectively make with any straight line lying in that plane are in a constant ratio.

CHAPTER II.

REFLECTION AND REFRACTION OF DIRECT PENCILS.

18. IF we consider a pencil as made up of an infinite number of rays all proceeding from a common point, it is clear that if such a pencil moving in one medium be incident on the surface of a second medium, it would be theoretically possible to calculate the direction of the refracted and reflected rays corresponding to each incident ray, and by considering the assemblage of these rays to obtain an idea of the form and position of the reflected and refracted pencils. The difficulties of calculation are however too great to allow this ordinarily to be done; and we have to content ourselves with approximations.

Approximate results can in all practical cases be obtained, so near to the truth that they represent the observed phenomena as accurately as the eye can discern them.

In practice the only surfaces at which reflection or refraction takes place in optical instruments are plane or spherical surfaces, or small portions of surfaces of revolution which are symmetrical with respect to the axis of revolution. These last can be always considered to coincide with portions of a sphere which has the same curvature as the surface of revolution at its vertex.

We shall therefore only consider the cases of plane and spherical surfaces.

19. In one case of great importance the accurate form and position of the *reflected* pencil can be obtained;

cases is sometimes determined by the eye which finally receives it, but more frequently, in the case of any complicated series of refractions and reflections, by the size of some one or other of the reflecting or refracting surfaces on which the light falls.

In all cases which we shall have to consider, the pencils will be very slightly divergent, that is, the solid angles of the cones of light considered will be very small.

22. The central ray of the pencil, or more strictly the geometrical axis of the cone of light, is called the axis of the pencil.

In any case of reflection or refraction, when the axis of the pencil coincides with the normal to the refracting or reflecting surface at the point of incidence of the axis, the pencil is said to be *directly incident* on the surface.

If the axis of the pencil do not coincide with the normal to the surface at the point of incidence, the pencil is said to be *obliquely incident.*

Thus, the axis of the pencil by which the point A is seen in the figure of Art. 20, is obliquely incident on the mirror CD.

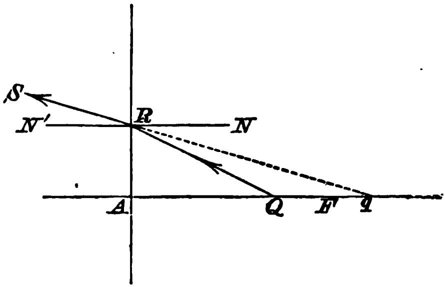
23. We shall first consider the effect of reflection and refraction when a small pencil is *directly* incident on a plane or spherical surface.

The first case, that of a pencil of any size reflected at a plane surface has been discussed in Art. 19. The reflected rays will form a cone whose vertex is at a point as far behind the mirror as the original point is in front of it.

24. Secondly, we have to consider the case of a pencil of light whose axis is *directly* incident on a plane surface capable of *refracting* light.

Let AR be a portion of the refracting surface, QA the axis of the incident pencil, QR any other ray of the

pencil; NRN' the normal to the surface at R, which is of course parallel to AQ.



Let μ be the index of refraction from the first medium into the second. Then by the law of refraction, if RS be the refracted ray, we have

$$\sin QRN = \mu \sin SRN',$$

or $$\sin RQA = \mu \sin RqA,$$

if SR produced meet AQ in q,

or $$\frac{AR}{RQ} = \mu \cdot \frac{AR}{Rq},$$

$$\therefore \quad Rq = \mu \cdot RQ \quad\text{.....................}\ (1).$$

This equation gives a relation by which the distance from A of the point where the refracted ray produced backwards meets the axis of the incident pencil can be determined. It is quite clear that the axis QA of the incident pencil is also the axis of the refracted pencil, since the refracted rays will be symmetrically placed with respect to this line.

It can be shown by means of equation (1) that if the position of R change, that of q will also change. Thus the refracted rays do not all accurately pass through one

point; and the refracted pencil is not accurately of a conical shape.

Nevertheless, if the original pencil be small, as is the case in all, or nearly all, the pencils we have practically to employ in optical instruments; by supposing AR very small, we shall find a limiting position of q, such that the rays of the refracted pencil will all very nearly pass through it, and which may thus be considered as approximately the vertex of the refracted pencil.

If we make R to approach indefinitely near to A, and F to be the limiting position of q, we get from (1)

$$AF = \mu \, . \, AQ.$$

AQ is usually denoted by the letter u, and AF by the letter v: this equation then becomes

$$v = \mu u \text{.................} (2).$$

25. In any case of direct refraction or reflection the axis of the incident pencil is also the axis of the reflected or refracted pencil.

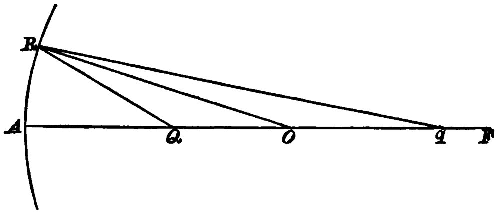
The point which may be considered as approximately the vertex of the reflected or refracted pencil is obtained as in the last Article, and is called the *geometrical focus* of the pencil after refraction or reflection.

The geometrical focus of a pencil after direct refraction or reflection may be defined as the limiting position of the point of intersection of any refracted or reflected ray with the axis of the pencil, when the point of incidence of the ray in question approaches indefinitely near to the point of incidence of the axis.

26. We have next to examine the position of the geometrical focus of a pencil *directly reflected* at *a spherical* surface.

Let QA be the axis of a pencil directly incident on a spherical reflecting surface. The incidence being direct, QA is the normal at A, and O the centre of the sphere must consequently lie in AQ.

Let QR be any ray of the pencil, incident at R. Join OR, and draw Rq so as to make the angle ORq equal to



the angle ORQ, then it is clear that Rq will be the reflected ray.

As R approaches A, the point q, in which Rq cuts AQ, will assume some limiting position. Let this be F. Then F is the geometrical focus whose position is required.

Since the angle ORQ is equal to the angle ORq we have by Euclid VI. 3,

$$QO : Oq :: QR : Rq,$$

or in the limit when R approaches indefinitely near to A, and consequently q to F,

$$QO : OF :: QA : AF \quad \text{.................} (1).$$

Let $\quad AO = r, \quad AQ = u, \quad AF = v.$

Then from (1) $\quad r - u : v - r :: u : v,$

$$\therefore \quad (r-u)v = (v-r)u,$$

$$\therefore \quad rv + ru = 2uv,$$

or dividing by urv,

$$\frac{1}{v} + \frac{1}{u} = \frac{2}{r} \quad \text{..........................} (2).$$

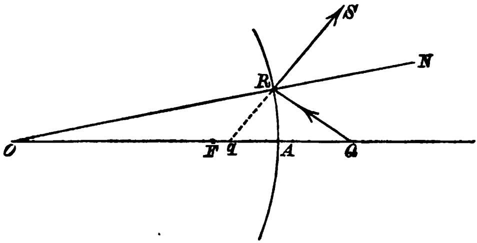
We have drawn the mirror concave, and have supposed the points Q and q to be on the same side of A as the centre of the surface.

It will be found that the formula (2) will be universally true whatever may be the relative positions of the points

A, Q, O and F, if the convention be adopted that lines measured in one direction from A, say to the right of A, shall be considered positive, and lines measured from A in the opposite direction shall be considered negative.

27. For instance, if the mirror be convex as in the figure, we still have from the law of reflection, by Euclid, VI. A,

$$QO : Oq :: QR : Rq,$$



or in the limit

$$QO : OF :: QA : AF,$$

and writing $-r$ for OA, $+u$ for AQ, and $-v$ for AF, we have

$$u-r : v-r :: u : -v,$$

$$\therefore \quad v(r-u) = u(v-r),$$

as before.

The student may draw other figures for himself, and verify that in every case the formula (2) of the last Article holds.

28. It is sometimes convenient to take the centre of the spherical surface as point of reference. In this case we shall take the figure of the last Article as the typical one, because in that figure all the distances involved are measured from O in one direction, and may be considered positive.

Let $OA = r$, $OQ = p$, $OF = q$.

Then since

$$QO : OF :: QA : AF,$$

we have

$$p : q :: p - r : r - q,$$

whence $$p(r - q) = q(p - r),$$

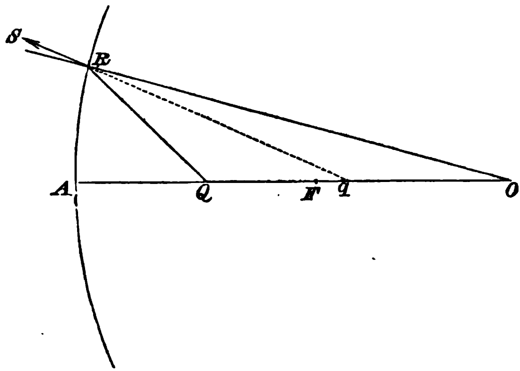
or $$pr + qr = 2pq,$$

whence $$\frac{1}{p} + \frac{1}{q} = \frac{2}{r}.$$

This formula, like that in Art. 26, can be adapted to all cases by the convention with regard to signs explained in Art. 26.

29. We have finally to investigate the position of the geometrical focus of a pencil directly incident on a *spherical refracting surface.*

Let O be the centre of the sphere, Q the vertex of the cone of light whose axis is incident at A. Since the incidence is direct, QA coincides with OA.



Let QR be any ray of the pencil, incident at R, and RS the corresponding refracted ray. Produce RS backwards.

to meet AQ in q, and let F be the limiting position of q when R approaches very near to A.

By the law of refraction, if μ be the refractive index,

$$\sin QRO = \mu \sin qRO,$$

$$\therefore \quad \frac{QO}{QR} \cdot \sin QOR = \mu \cdot \frac{qO}{qR} \sin qOR,$$

or $$QO \cdot qR = \mu \cdot qO \cdot QR.$$

Hence in the limit when R approaches very near to A, and q to F,

$$QO \cdot FA = \mu\, FO \cdot QA,$$

whence if $$AO = r,\ AQ = u,\ AF = v,$$

$$(r-u)v = \mu(r-v)\,u,$$

$$\therefore\ \mu ru - rv = (\mu - 1)\,uv,$$

or dividing by urv

$$\frac{\mu}{v} - \frac{1}{u} = \frac{\mu - 1}{r}.$$

30. The formula proved in the last Article is of fundamental importance. It virtually contains all that have been given before.

Thus we may consider a plane surface as the limiting form of a sphere whose radius is infinite; and thus by making r infinite, we shall get the geometrical focus for a pencil of light directly incident on a *plane* refracting surface. The formula of the last Article becomes in this case,

$$\frac{\mu}{v} - \frac{1}{u} = 0,$$

or $$v = \mu u,$$

the formula of Art. 24.

Again it is found, and the reason will be given in the next Article, that any formula for refraction will give the corresponding formula for reflection, by giving to μ the value -1. Thus the formula for the geometrical focus of a pencil of light, directly reflected at a spherical surface,

will be obtained from that of the last Article by this substitution. The formula becomes

$$-\frac{1}{v}-\frac{1}{u}=-\frac{2}{r},$$

or $$\frac{1}{v}+\frac{1}{u}=\frac{2}{r},$$

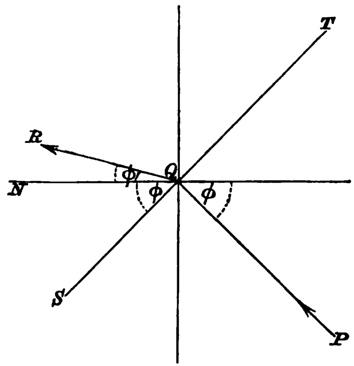
the same as in Art. 26.

For reflection at a plane surface we either make r infinite in this last formula, or put μ equal to -1 in the formula $v=\mu u$. Either of these methods gives

$$v=-u,$$

agreeing with the geometrical result of Arts. 19 and 23.

31. The derivation of formulæ for reflection from corresponding formulæ for refraction by the substitution of -1 for μ can be explained by the following considerations.



Let PQ be any ray incident on a refracting surface at Q, and let QR be the corresponding refracted ray. Let QN be the normal drawn internally.

Then by the law of refraction QR is drawn *above* QN at an angle ϕ' determined by the equation

$$\sin\phi' = \frac{1}{\mu}\sin\phi \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where ϕ is the angle of incidence.

In this equation, put $\mu = -1$,

$$\therefore \sin\phi' = -\sin\phi,$$

$$\therefore \phi' = -\phi.$$

Now as a *positive* value of ϕ' indicates a line drawn at an angle ϕ' *above* QN, a *negative* value of ϕ' will indicate a line drawn *below* QN. Hence consistently with the usual geometrical interpretation of positive and negative, the line given geometrically by the equation (1) when $\mu = -1$ is a line QS below QN, inclined to QN at an angle equal to the angle of incidence, that is, the equation in that case determines the direction of the reflected ray produced. Hence all formulæ which give the point of intersection of a refracted ray with a given line, will determine the corresponding point in the case of a reflected ray by the substitution $\mu = -1$.

32. We have now determined the position of the vertex of a pencil of light after refraction or reflection, when directly incident on a plane or spherical surface.

Before proceeding to examine the case of obliquely incident pencils, we must consider a little farther the formulæ of Arts. 26 and 29.

33. Taking the formula

$$\frac{1}{v} + \frac{1}{u} = \frac{2}{r},$$

which connects the distances of Q and F from A in the case of a reflecting spherical surface, we see first that the points Q and F are interchangeable, that is, if Q move to where F is at any time, F will move to where Q was previously.

The points Q and F are sometimes called *conjugate foci*.

When the original pencil consists of parallel rays Q is at an indefinitely great distance and $\frac{1}{u}=0$. Hence AF or $v=\frac{r}{2}$.

This position of the point F is sometimes called the *principal focus* of the reflecting surface.

Similarly from the formula,

$$\frac{\mu}{v}-\frac{1}{u}=\frac{\mu-1}{r},$$

putting $u=\infty$, or $\frac{1}{u}=0$, we get $v=\frac{\mu r}{\mu-1}$.

The point F whose distance from A is thus determined is called the principal focus of the refracting surface.

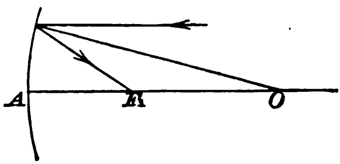
34. Again, taking the formula

$$\frac{1}{v}+\frac{1}{u}=\frac{2}{r},$$

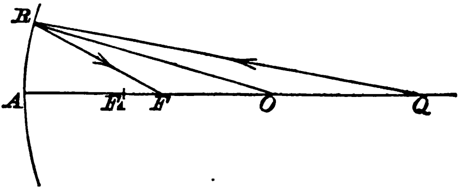
it is easy to trace out the changes in the position of F corresponding to various positions of Q.

First we may notice that if u increases, v must decrease, since $\frac{1}{v}+\frac{1}{u}$ remains of invariable value. Hence Q and F move in opposite directions.

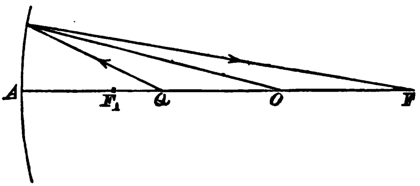
When Q is at an infinite distance to the right hand, $v=\frac{r}{2}$, and hence F is at a point F_1 half-way between A and O.



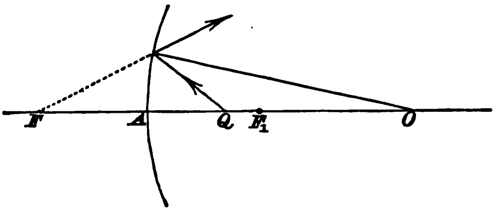
As Q moves up towards O, F also moves towards O, and at O they coincide; for when $u=r$, $v=r$.



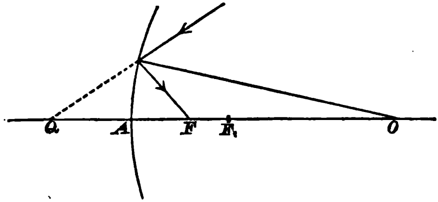
When Q moves from O to F_1, F moves from O to a great distance to the right, and when Q is at F_1, $u=\frac{r}{2}$, $\therefore v=\infty$, and the reflected rays are parallel.



When Q is between F_1 and A, F moves up from a great distance to the left towards A, v being negative, since $u<\frac{r}{2}$, and consequently $\frac{1}{u}>\frac{2}{r}$.



When Q is indefinitely near to A, $\frac{1}{u}$ is very large and positive, and therefore $\frac{1}{v}$ must be very large and negative. Thus F must approach indefinitely near to A, and when Q reaches A, F also reaches A.



When Q is to the left of A (which can only be the case when, by some artificial process of refraction or reflection, we have made a pencil of light converge to a point behind A), u is negative, and v will be positive and $<\frac{r}{2}$. Hence F lies between A and F_1. As Q moves farther to the left, F approaches nearer to F_1, till when Q has got to a point at an infinite distance to the left of A, $u=\infty$, and F coincides again with F_1.

35. From the formula

$$\frac{\mu}{v}-\frac{1}{u}=\frac{\mu-1}{r},$$

we see that in the case of refraction at a spherical surface u and v must increase or decrease together. Hence Q and F move in the same direction.

The student can easily verify the following facts.

(1) When Q is at an infinite distance to the right of A F is at a point F_1 such that $AF_1=\frac{\mu r}{\mu-1}$.

(2) As Q moves from an infinite distance up to O, F moves from F_1 to O, and at O they coincide.

(3) As Q moves from O to A, F also moves from O to A, being nearer to O than Q is, and at A, Q and F again coincide.

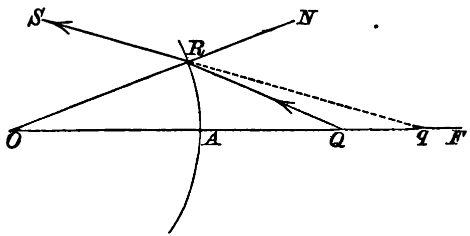
(4) As Q moves from A to a point F_2 at a distance $\frac{r}{\mu-1}$ to the left of A, F moves from A to an infinite distance to the left of A.

(5) As Q moves from F_2 to an infinite distance to the left of A, F moves from an infinite distance to the right of A up to the point F_1 previously mentioned.

36. In the two previous Articles we have supposed the spherical surfaces concave to the right. The student can examine the cases in which they are concave to the left, and r consequently negative.

37. It is sometimes convenient to replace the formula of Art. 29 by a formula giving the distances of the conjugate foci from the centre of the surface instead of from the point A. In this case we shall draw a figure with the spherical surface concave to the left, in order that all the lines may be positive.

Let QR be any ray of the pencil, incident at R, RS, produced backwards to meet the axis in q, the correspond-



ing refracted ray. Let F be the limiting position of q when R comes to A.

Then as before

$$\sin QRN = \mu \sin qRN,$$

$$\therefore \frac{QO}{QR} \sin QOR = \mu \frac{qO}{qR} \sin qOR,$$

$$\therefore QO \,.\, qR = \mu \,.\, qO \,.\, QR,$$

or in the limit

$$QO \,.\, FA = \mu \,.\, FO \,.\, QA.$$

Let $OQ = p$, $OA = r$, $OF = q$,

$$\therefore p(q - r) = \mu q(p - r),$$

whence

$$\frac{\mu}{p} - \frac{1}{q} = \frac{\mu - 1}{r}.$$

This formula may be instructively compared with the formula

$$\frac{\mu}{v} - \frac{1}{u} = \frac{\mu - 1}{r},$$

of Art. 29.

The two formulæ are identical in form, p and q being replaced by v and u; an alliterative way of remembering the substitution. In either case, of the two letters involved, the first in alphabetical order refers to the incident pencil, and the other to the refracted pencil. Thus if the student remember either formula accurately, he will easily obtain the other.

EXAMPLES ON CHAPTER II.

1. A stick partly immersed in water appears bent upwards at the point where it meets the water. Explain this.

2. A stick is partly immersed in water, being inclined to the horizon at an angle whose tangent is $\frac{4}{3\sqrt{3}}$. Find at what angle the part which is under water will appear to be inclined to the horizon, to an eye placed at some distance vertically above the stick; the refractive index of water being $\frac{4}{3}$.

3. A pencil of light is incident directly on a refracting sphere of radius a, whose refractive index is $\frac{4}{3}$. Find the position of the geometrical focus of the refracted pencil, the origin of light being at a distance of $4a$ from the centre of the sphere.

4. A speck at the back of a plate of glass, one inch thick, is looked at by an eye placed just in front of the plate. Find at what distance the eye will imagine the speck to be, the refractive index of glass being $\frac{3}{2}$.

5. A candle is placed in front of a concave spherical mirror, whose radius is one foot, at a distance of 5 inches from the mirror. Where will the image of the candle appear to be to an eye situated in the axis of the mirror?

6. If the candle in the last question be moved to a position two inches farther from the mirror, how will the position of the image be changed?

7. Show that if the velocity with which light travels in any medium were directly proportional to the refractive index of that medium, the time occupied by light, which is refracted directly at a plane surface, in reaching any point in the second medium would be the same as it would occupy in travelling all the distance from the geometrical focus after refraction, to the same point, in a medium of the same nature as the second medium.

8. F, Q are conjugate foci of a mirror whose centre is O and radius OA. Prove that if any point P be joined to the four points A, F, O, Q, and a straight line $afoq$ be drawn to cut these lines in a, f, o, q, then f, q are conjugate foci of a mirror whose centre is o and radius oa.

9. If q be the geometrical focus of a pencil of light after reflection at a spherical surface, whose centre is C, corresponding to a luminous point at Q, and F be the principal focus, prove that $FC^2 = FQ \cdot Fq$.

10. The locus of the image of a luminous point reflected in a plane mirror is a circle. Prove that the mirror always touches a conic section or passes through a fixed point.

11. A luminous point is placed in front of a plane reflecting surface. If this surface turn in any manner about a point in its own plane, prove that the geometrical focus of the rays after reflection lies on a sphere.

Prove that this will also be the case if the plane mirror move so as always to touch a prolate spheroid of which the luminous point is one focus.

12. A luminous point is placed in front of a refracting medium bounded by a transparent plane surface. Prove that if the bounding plane move in any manner about a fixed point in itself, the geometrical focus of the rays after refraction into the medium always lies on the surface of a sphere.

13. Three plane mirrors are all perpendicular to a given plane. Show that if a luminous point be placed anywhere on the circumference of the circle which is described round the triangle formed by the intersections of the mirrors with the given plane, the three images of the point formed by one reflection at each mirror respectively will all lie in a straight line.

14. Four plane mirrors are all perpendicular to a given plane. Find the position of a luminous point that its images formed by one reflection at each mirror respectively may all lie in a straight line.

15. A luminous point is placed within a polygon whose sides are reflecting surfaces: if the image of the point, formed by reflection at each side coincide with the point of intersection of the two adjacent sides, prove that the polygon is a regular hexagon, and the luminous point at its centre.

16. If a pencil of diverging rays incident on a convex spherical surface, is refracted to a point as far behind the surface as the origin of light is in front of it, show that the radius of the surface is $\frac{\mu-1}{\mu+1}$ of the distance of the point of light from the surface.

17. If any circle be drawn through two conjugate foci in the case of a spherical reflecting surface; prove that, in general, two other conjugate foci lie on the same circle.

18. A luminous point is placed within a reflecting sphere: prove that its distance from the centre is a harmonic mean between the distances, from the centre, of the geometrical foci after reflection at the opposite portions of the surface.

CHAPTER III.

REFLECTION AND REFRACTION OF OBLIQUE PENCILS.

38. WE have now discussed all the cases of direct incidence, and have ascertained that after direct refraction, or reflection, the form of a pencil of light, originally conical, is still approximately conical.

We shall find that this is not the case when a small pencil is obliquely incident; but that the rays of such a pencil, after refraction or reflection, all pass approximately through two straight lines, at right angles to each other and to the axis of the refracted or reflected pencil.

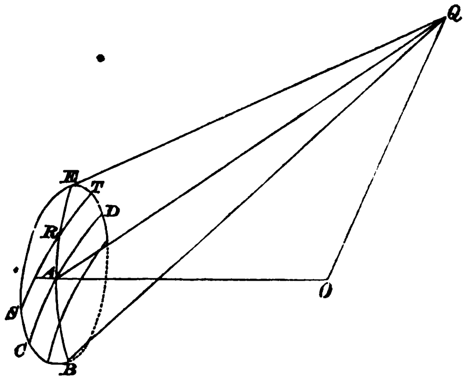
39. In the previous chapter we only considered those rays of the pencil which lie in one plane through its axis; but the pencil and the surface being originally symmetrical with respect to this axis, the geometrical foci of the rays in all such planes will be the same.

In fact, if with vertex Q and QA as axis, in any of the cases discussed in the last chapter, we describe a cone with semi-vertical angle RQA, the rays of the original pencil which lie on this cone, after refraction or reflection, will all cut the line AQ in the same point q.

40. Thus, let Q be the vertex and QA the axis of a pencil obliquely incident at A on a plane or spherical refracting or reflecting surface $BCDE$.

Let QO be that normal to this surface which passes through Q, which in the case of a spherical surface will be

the line joining Q with the centre of the sphere. Let AO be the normal at A.

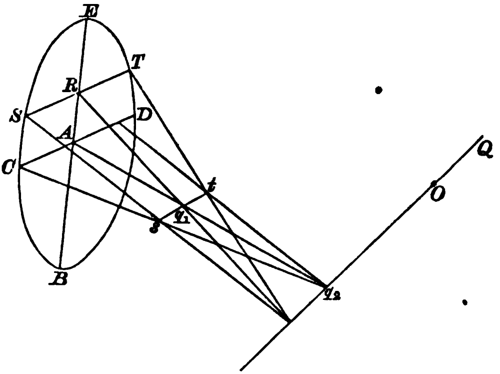


With vertex Q and axis QO, and semi-vertical angle AQO, describe a conical surface. This surface will cut the refracting or reflecting surface in a curve CAD; and by the remark of the last article, all the rays in the pencil incident on CAD will pass through the same point of QO after refraction or reflection.

If with vertex Q and axis QO, and semi-vertical angle slightly differing from AQO, we describe a cone, this cone will cut the surface $BCDE$ in a curve SRT, and all the rays incident at points of this curve will, after refraction, pass through some one point in the line QO.

If the lengths of CD, BE be small compared with AQ and AO, which is always the case in practice, the lines CAD, SRT will be very nearly straight lines; and we may consider the set of rays which fall on the points of CAD after refraction to lie in a plane which passes through CD and the point in QO where they all meet.

Similarly, we may consider the set of rays which fall on SRT, after refraction or reflection, to lie in another plane, which passes through SRT and the point in QO where all these rays meet.



These two planes are evidently each at right angles to the plane QOA, and will intersect in a straight line $sq_1 t$, which is also perpendicular to the plane QOA.

If we suppose SRT to approach indefinitely near to CAD, this line will assume some limiting position; and in this limiting position, if the pencil be small, it is a straight line through some point of which every ray of the refracted or reflected pencil will nearly pass.

This limiting position of the line is called the *primary focal line;* and the point q_1, where it cuts the plane QOA, is called the primary focus.

The plane QOA is called the *primary plane*, and the point where the primary focal line cuts the primary plane may be evidently considered as a focus, or point of concentration, of the refracted or reflected rays which lie in the primary plane.

41. Again, if we consider the rays incident on CAD, these rays, after refraction or reflection, all pass through the point q_2, where the axis of the refracted or reflected pencil cuts the line QO. This point q_2 may be considered as the focus of all rays in the section of the reflected or

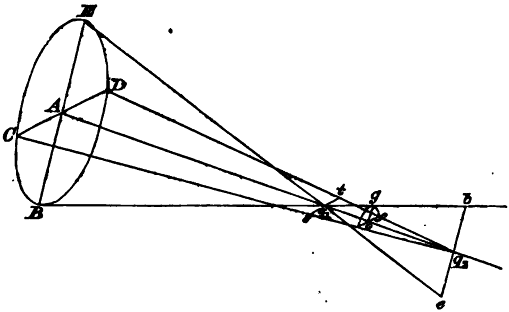
refracted pencil by the plane through CAD and the axis of this pencil.

This point q_2 is called the *secondary focus*, and the plane through CAD and Aq_2 is sometimes called the *secondary plane* of the refracted or reflected pencil.

It will be pretty clear that if we take a section of the pencil by a plane through q_2 parallel to the tangent plane to the refracting or reflecting surface at A, this section will be a sort of figure of eight, having no width at the point q_2, but swelling out above, because the rays have not come together to meet in QO, and swelling out below, because the rays having passed through QO have again begun to diverge.

This figure of eight is nearly a straight line, the width of the loops being small, if the pencil is small. It is called the secondary focal line.

42. We see thus that in determining the form of a pencil after oblique refraction or reflection we have to determine two foci, and that the pencil is not approximately of a conical shape.



If we take a series of sections of the resulting pencil by planes parallel to the tangent plane to the refracting or reflecting surface at A, we see that the section at q_1 is a straight line perpendicular to the primary plane, while that

at q_2 is a straight line in the primary plane. As the cutting plane passes from q_1 to q_2, the section is of an oval shape, first, when near q_1, having its longest diameter perpendicular to the primary plane; and when near to q_2, having its greater length in the primary plane.

There must therefore be some position of the cutting plane between q_1 and q_2, for which the diameters of the section in, and perpendicular to the primary plane, are equal. In this position the section will be approximately circular, and it is called the *circle of least confusion.*

It is probably the nearest approach to *a point* of any section of the pencil, and is thus the point where any eye receiving the obliquely refracted or reflected pencil would consider the image of the original point of light to be placed.

Its position can be thus determined when those of the two foci are known.

Let the diameters of the surface $ABCDE$, in and perpendicular to the primary plane, be $2\lambda_1$ and $2\lambda_2$ respectively, that is, let $AD=\lambda_2$, $AB=\lambda_1$.

Also let $Aq_1=v_1$ and $Aq_2=v_2$, q_1 and q_2 being the primary and secondary foci.

Let o be the point where the required section cuts Aq_1q_2, and let $Ao=x$.

Let of, og be the radii of the circle of least confusion, perpendicular to the primary plane, and in that plane respectively:

$\therefore$ $of=og$ by definition of circle of least confusion.

Again, by similar triangles,

$$of : AD :: oq_2 : Aq_2;$$

$$\therefore\ of=\lambda_2\frac{v_2-x}{v_2}.$$

Similarly, $$og : AB :: oq_1 : Aq_1;$$

$$\therefore\ og=\lambda_1\frac{x-v_1}{v_1}.$$

But, $$og = of;$$

$$\therefore \lambda_1 \frac{x - v_1}{v_1} = \lambda_2 \frac{v_2 - x}{v_2};$$

$$\therefore \left(\frac{\lambda_1}{v_1} + \frac{\lambda_2}{v_2}\right) x = \lambda_1 + \lambda_2;$$

$$\therefore \frac{\lambda_1}{v} + \frac{\lambda_2}{v_2} = \frac{\lambda_1 + \lambda_2}{x} \quad \text{.................. (1).}$$

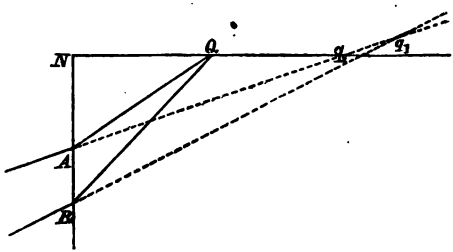
The position of o is thus determined if those of q_1 and q_2 are known.

If, as is usually the case, the surface *ABCDE* has a circular boundary $\lambda_1 = \lambda_2$, and the equation (1) becomes

$$\frac{1}{v_1} + \frac{1}{v_2} = \frac{2}{x} \quad \text{........................ (2).}$$

43. We have now to determine the position of the primary and secondary foci in the three cases of a pencil obliquely incident on a plane refracting surface and on a spherical refracting and reflecting surface respectively. A pencil obliquely incident on a plane reflecting surface is as we know reflected exactly from a point.

44. Let QA be the axis of a pencil incident at A on a plane refracting surface, q_2A the direction of the refracted



ray produced backwards to meet a line QN, drawn through Q perpendicular to the plane surface, in q_2—the secondary focus of the refracted pencil.

Let QR be any other ray of the pencil in the primary plane, and let the corresponding refracted ray produced backwards cut Aq_2 in q_1. The limiting position of q_1 when R approaches A is the primary focus of the refracted pencil.

$$\text{Let } AQ=u, \quad Aq_1=v_1, \quad Aq_2=v_2,$$

and let μ be the refractive index from the first medium into the second. Let the angle of incidence of QA be denoted by ϕ, and the corresponding angle of refraction be ϕ'.

$$\text{Then } AQN=\phi, \quad Aq_2N=\phi'.$$

And by the law of refraction,

$$\sin\phi=\mu\sin\phi';$$

$$\therefore \frac{AN}{AQ}=\mu\cdot\frac{AN}{Aq_2};$$

$$\therefore Aq_2=\mu\cdot AQ,$$

$$v_2=\mu u \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Again, let ϕ_1, ϕ_1' be the angles of incidence and refraction of the ray QR;

$$\therefore QRA=90^\circ-\phi_1, \quad q_1RA=90^\circ-\phi_1';$$

$$\therefore \frac{AR}{AQ}=\frac{\sin AQR}{\sin ARQ}=\frac{\sin(\phi_1-\phi)}{\cos\phi_1},$$

$$\frac{AR}{Aq_1}=\frac{\sin Aq_1R}{\sin ARq_1}=\frac{\sin(\phi_1'-\phi')}{\cos\phi_1'};$$

$$\therefore \frac{Aq_1}{AQ}=\frac{\cos\phi_1'}{\cos\phi_1}\cdot\frac{\sin(\phi_1-\phi)}{\sin(\phi_1'-\phi')}$$

$$=\frac{\cos\phi_1'\,.\sin\frac{1}{2}(\phi_1-\phi)\,.\cos\frac{1}{2}(\phi_1-\phi)}{\cos\phi_1\,.\sin\frac{1}{2}(\phi_1'-\phi')\,.\cos\frac{1}{2}(\phi_1'-\phi')}.$$

$$\text{Now } \sin\phi=\mu\sin\phi',$$

$$\sin\phi_1=\mu\sin\phi_1';$$

$$\therefore (\sin\phi_1-\sin\phi)=\mu(\sin\phi_1'-\sin\phi');$$

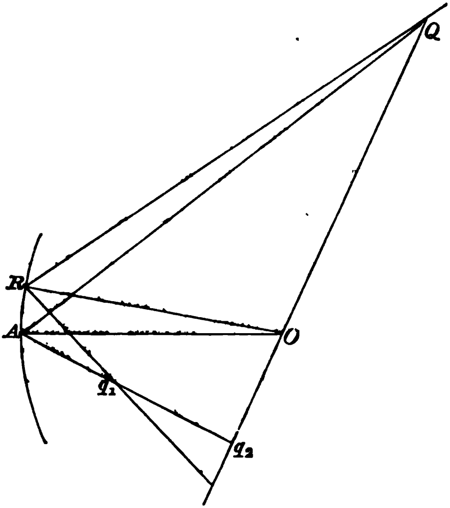
$$\therefore \frac{\sin\frac{1}{2}(\phi_1-\phi)}{\sin\frac{1}{2}(\phi_1'-\phi')}=\mu\cdot\frac{\cos\frac{1}{2}(\phi_1'+\phi')}{\cos\frac{1}{2}(\phi_1+\phi)};$$

$$\therefore \frac{Aq_1}{AQ}=\frac{\cos\phi_1'}{\cos\phi_1}\cdot\frac{\mu\cdot\cos\frac{1}{2}(\phi_1'+\phi')}{\cos\frac{1}{2}(\phi_1+\phi)}\cdot\frac{\cos\frac{1}{2}(\phi_1-\phi)}{\cos\frac{1}{2}(\phi_1'-\phi)}.$$

But in the limit when R comes up to A, we have $\phi_1=\phi$ and $\phi_1'=\phi'$; and this formula becomes

$$v_1=\mu\cdot\frac{\cos^2\phi'}{\cos^2\phi}\cdot u\ \ldots\ldots\ldots\ldots\ldots\ldots\ (2).$$

45. Secondly, let QA be the axis of a pencil obliquely incident on a spherical reflecting surface. Let Aq_2 the cor-



responding reflected ray cut QO, the line joining Q with the centre of the sphere in q_2, which is the secondary focus.

Let QR be any other ray in the primary plane, and let Rq_1 be the corresponding reflected ray, cutting Aq_2 in a

point q_1. The limiting position of q_1 when R is very near to A, will be the primary focus.

Let $AQ=u$, $Aq_1=v_1$, $Aq_2=v_2$, $AO=r$.

Let $QAO=\phi$ the angle of incidence of the axis;

$\therefore$ $q_2AO=\phi$ the angle of reflection of the axis.

Let each of the angles QRO, q_1RO be called ϕ_1.

Then we have

$$\Delta QAq_2=\Delta QAO+\Delta q_2AO;$$

$$\therefore \tfrac{1}{2}uv_2 \sin 2\phi=\tfrac{1}{2}ur \sin\phi+\tfrac{1}{2}v_2r\sin\phi;$$

$$\therefore 2uv_2\cos\phi=ur+v_2r;$$

$$\therefore \frac{2\cos\phi}{r}=\frac{1}{v_2}+\frac{1}{u} \quad\text{..................(1).}$$

Again,
$$\angle AOq_2=\phi+\angle AQO,$$
$$\angle ROq_2=\phi_1+RQO;$$
$$\therefore \angle AOR=\phi_1-\phi+\angle RQA.$$

Similarly,
$$\angle Rq_1A=\phi_1-\phi+\angle ROA;$$
$$\therefore 2\angle ROA=\angle Rq_1A+\angle RQA\quad\text{.........(2).}$$

But all the three angles ROA, Rq_1A, RQA, are ultimately very small, and their circular measures may be replaced by their sines. Also we may consider RA as a portion of a straight line at right angles to AO.

Hence, from the triangle RQA,

$$\sin RQA=\frac{RA}{RQ}\sin RAQ=\frac{RA}{u}\,.\cos\phi,$$

and from the triangle Rq_1A,

$$\sin Rq_1A=\frac{RA}{Rq_1}\sin RAq_1=\frac{RA}{v_1}\cos\phi,$$

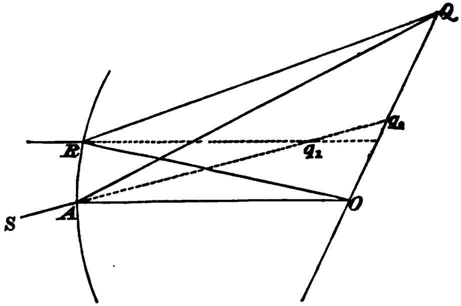
and $ROA=\dfrac{RA}{r}$;

whence, substituting in the formula (2) we get dividing by RA,

$$\frac{2}{r} = \frac{\cos\phi}{u} + \frac{\cos\phi}{v_1},$$

or
$$\frac{1}{v_1} + \frac{1}{u} = \frac{2}{r\cos\phi} \quad \text{.....................(3).}$$

46. Thirdly, let QA be the axis of a pencil obliquely incident on a spherical refracting surface, AS the corre-



sponding refracted ray, produced backwards if necessary to cut QO in q_2. Let QR be another ray in the primary plane and let the line of the corresponding refracted ray cut q_2A in q_1.

The limiting position of q_1 is the primary focus, and q_2 is the secondary focus.

Let
$$QAO = \phi, \quad q_2AO = \phi',$$
$$QRO = \phi_1, \quad q_1RO = \phi_1',$$
$$AQ = u, \quad Aq_1 = v_1, \quad Aq_2 = v_2, \quad AO = r.$$

Then, as in the last Article,

$$ROA = Rq_1A + \phi' - \phi_1',$$
$$ROA = RQA + \phi - \phi_1,$$

and as in the last Article,

$$\angle ROA = \frac{AR}{r}, \quad Rq_1A = \frac{AR\cos\phi'}{v_1}, \quad RQA = \frac{AR\cos\phi}{u};$$

$$\therefore \frac{AR}{r} - \frac{AR\cos\phi'}{v_1} = \phi' - \phi_1' \;\ldots\ldots\ldots\; (1).$$

$$\frac{AR}{r} - \frac{AR\cos\phi}{u} = \phi - \phi_1 \;\ldots\ldots\ldots\ldots\; (2).$$

Also, as in Art. 44,

$$\sin\phi = \mu\sin\phi', \quad \sin\phi_1 = \mu\sin\phi_1',$$

whence $$\frac{\sin\frac{1}{2}(\phi-\phi_1)}{\sin\frac{1}{2}(\phi'-\phi_1')} = \mu \cdot \frac{\cos\frac{1}{2}(\phi'+\phi_1')}{\cos\frac{1}{2}(\phi+\phi_1)},$$

or since in the limit $\phi - \phi_1$ and $\phi' - \phi_1'$ are indefinitely small, replacing sines of small angles by their circular measures, ultimately,

$$\frac{\phi-\phi_1}{\phi'-\phi_1'} = \frac{\mu\cos\phi'}{\cos\phi},$$

or $$(\phi-\phi_1)\cos\phi = (\phi'-\phi_1')\,\mu\cos\phi',$$

whence from (1) and (2),

$$\mu\cos\phi'\left(\frac{AR}{r} - \frac{AR\cos\phi'}{v_1}\right) = \cos\phi\left(\frac{AR}{r} - \frac{AR\cos\phi}{u}\right);$$

$$\frac{\mu\cos^2\phi'}{v_1} - \frac{\cos^2\phi}{u} = \frac{\mu\cos\phi' - \cos\phi}{r} \;\ldots\ldots\ldots\; (3).$$

Again, $$\Delta QAq_2 = \Delta QAO - \Delta q_2AO;$$

$$\therefore \tfrac{1}{2}uv_2\sin(\phi-\phi') = \tfrac{1}{2}ur\sin\phi - \tfrac{1}{2}v_2r\sin\phi';$$

$$\therefore \frac{\sin\phi\,.\cos\phi' - \cos\phi\sin\phi'}{r} = \frac{\sin\phi}{v_2} - \frac{\sin\phi'}{u};$$

and substituting for $\sin\phi$ its value $\mu\sin\phi'$ and dividing throughout by $\sin\phi'$, we have

$$\frac{\mu\cos\phi' - \cos\phi}{r} = \frac{\mu}{v_2} - \frac{1}{u} \;\ldots\ldots\ldots\ldots\; (4).$$

47. We have now discussed to a certain order of approximation the alterations produced in a given small pencil of light by one refraction or one reflection, whether the incidence be oblique or direct. In the three succeeding chapters we have to examine to the same order of approximation the effect produced on such pencils by a number of such reflections or refractions in certain important cases.

*The student may notice that the results of the last three Articles include the positions of the geometrical foci investigated in Chapter II. These latter may be obtained from the formulæ of this Chapter by giving to ϕ and ϕ' the value zero.

It may be mentioned here that although, in accordance with the remark of Art. 42, the circle of least confusion is probably the position at which the eye sees an object by a pencil which has been obliquely refracted or reflected, it is sometimes convenient to assume the primary or secondary focus as the position of the image.

If the obliquity be small these points will all be close together, and it will not matter much which of them we take.

EXAMPLES. CHAPTER III.

1. A small pencil of parallel rays is incident at an angle of 60^0 on a spherical reflecting surface. Find the position of the focal lines.

2. A small pencil of parallel rays is incident on a spherical refracting surface at an angle of 60°, the refractive index being $\sqrt{3}$. Find the position of the focal lines.

3. In each of the last two examples find the position of the circle of least confusion on the supposition that the incident pencil is a right circular cylinder.

4. The refractive index of a medium being $\frac{4}{3}$, find the position of the primary focus of a pencil incident on a sphere formed of that medium, at an angle whose cosine is $\frac{\sqrt{7}}{3\sqrt{3}}$.

5. A pencil is incident obliquely on a spherical refracting surface at an angle whose tangent is equal to the refractive index of the sphere. Find the position of the focal lines.

6. A small pencil diverges from one extremity of the diameter of a sphere whose interior surface reflects light, and is incident on the sphere, so that its axis after reflection passes through the other end of the same diameter. Find the position of the focal lines, and show that $v_2 = 3v_1$.

7. Find the position of the point, from which light must diverge so that after refraction at a sphere whose refractive index is μ the primary and secondary foci may coincide.

Show that the point must be at a distance μr from the centre of the sphere.

8. A small pencil of parallel rays is incident on a concave spherical reflector at an angle of $\frac{\pi}{4}$. Find the position of the focal lines and the circle of least confusion, assuming (1) the border of the mirror to be circular; (2) to be elliptical with its diameters in the primary and secondary planes in the ratio of $\sqrt{2}$ to 1.

9. If O be the origin of light, P the point of incidence of the axis, and if the perpendicular to OP through O meet the tangent at P, at the foot of the perpendicular from the secondary focus on the same tangent; prove that the primary focus is at an infinite distance.

10. A pencil of parallel rays is incident obliquely on a convex refracting spherical surface. Find the position of the primary and secondary focal lines. If the angle of incidence be $\frac{\pi}{3}$ and the primary focus be on the surface of the sphere, show that the angle of refraction is the complement of the critical angle.

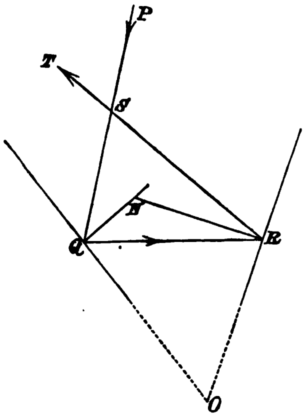
11. A small pencil diverges from a point in the surface of a spherical shell polished internally, and is twice reflected, show that if the normal at the first point of incidence pass through the final primary focus, the angle of incidence was $\frac{1}{2}\cos^{-1}\frac{3}{4}$.

CHAPTER IV.

ON REFLECTIONS AT TWO OR MORE PLANE SURFACES.

48. WE have first to prove that if a ray be reflected successively at two plane mirrors so that its

Fig. (1).

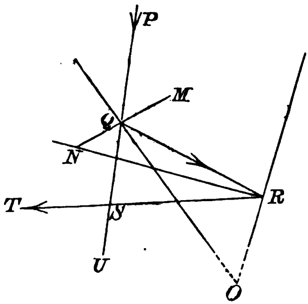


course throughout lies in a plane at right angles to each of them, its deviation from its original direction after two reflections will be double of the angle between the mirrors.

Let a ray be incident on one plane mirror at Q in the direction PQ, and let it be reflected along QR so as to

fall on a second mirror at R and be again reflected along RST.

Fig. (2).



Let PQ or PQ produced meet RT in S; then in figure (1) QST, and in figure (2) UST is the angle between the first direction of the ray of light and its last direction, that is, the deviation of the ray, and in either case this deviation is double of the angle POQ between the mirrors.

Let QN, RN be the normals to the mirrors at the points Q and R, meeting, produced if necessary, in N; QN and RN bisect the angles PQR and SRQ respectively.

Then in figure (1)

$$\begin{aligned} \angle QST &= \angle SRQ + \angle SQR \\ &= 2\angle NRQ + 2\angle NQR \\ &= 2(180^0 - QNR) \\ &= 2\angle ROQ. \end{aligned}$$

In figure (2)

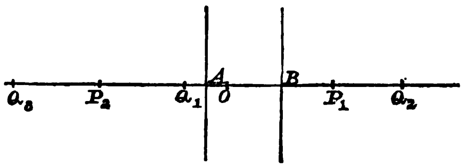
$$\begin{aligned} \angle UST &= \angle PSR \\ &= \angle PQR - \angle QRS \\ &= 2\angle MQR - 2\angle QRN \\ &= 2\angle QNR \\ &= 2\angle QOR. \end{aligned}$$

In either case the deviation is double the angle between the mirrors.

The property proved in this proposition is that on which the construction of the sextant depends. For a description of this instrument the reader is referred to any treatise on astronomy.

49. Again, let light emanate from a luminous point situated between two plane mirrors and after incidence on one of them be reflected so as to fall on the second, and then back again on to the first, and so on; the position and number of the images of the luminous point formed by successive reflections are determined by some very simple laws which we proceed to investigate.

Let the mirrors, in the first place, be parallel; and let O be a luminous point somewhere between them. Draw



AOB through O perpendicular to the planes of the mirrors, and produce it indefinitely in both directions.

Take $AQ_1=AO$, $BQ_2=BQ_1$, $AQ_3=AQ_2$,

and so on. Then, by Art. 19, Q_1, Q_2...are the geometrical foci of a pencil of light originally proceeding from O and reflected, first, by the surface A, then by the surface B, and so on, that is, they are the positions of the successive images of O, as seen by an eye placed anywhere between the mirrors, obtained by successive reflections, beginning with the surface A.

Similarly, if $BP_1=OB$, $AP_2=AP_1$, $BP_3=BP_2$, and so on; P_1, P_2, P_3... are the positions of the successive images of O, obtained by reflection at B and A alternately, beginning with B.

Let now $OA=a$, $OB=b$, $AB=c$, $\therefore c=a+b$.

Then $OQ_1=2a$,

$$OQ_2=BQ_2+BO=BQ_1+BO=2a+2b=2c,$$
$$OQ_3=AQ_3+AO=AQ_2+AO=2c+2a,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

And in this manner $OQ_{2n}=2nc$,

$$OQ_{2n+1}=2nc+2a.$$

Similarly

$$OP_1=2b,$$
$$OP_2=2c,$$
$$OP_3=2c+2b,$$
$$\dots\dots\dots\dots$$
$$OP_{2n}=2nc,$$
$$OP_{2n+1}=2nc+2b.$$

These formulæ give the positions of the successive images.

It is evident that Q_1, $Q_3 \dots Q_{2n+1}$ fall to the left of O, as also do $P_2 . P_4 \dots P_{2n}$; while Q_2, $Q_4 \dots Q_{2n}$, as also $P_1, P_3 \dots P_{2n+1}$, fall to the right of O.

It follows from these formulæ that

$$Q_1P_2=Q_3P_4=\dots=2c-2a=2b=OP_1,$$

and similarly

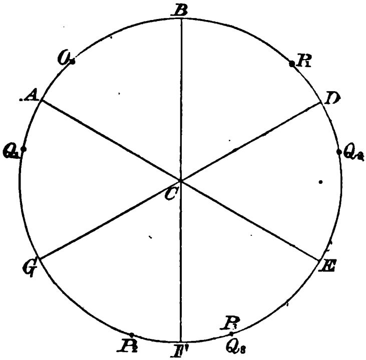
$$Q_2P_3=Q_4P_5=\dots\dots\dots\dots=2b=OP_1,$$

while the lengths P_2Q_3, $P_4Q_5 \dots P_1Q_2 \dots$ are all equal to OQ_1.

If the point O be midway between the mirrors $a=b$, and the successive images will all be arranged along the line AOB, each image being at a distance c from the nearest one.

50. Let the luminous point O be placed between two mirrors inclined at any angle. The images formed by successive reflections will no longer be arranged in a straight line, but will, as we shall see, all lie on a circle.

The figure represents sections of the mirrors by a plane passing through the luminous point perpendicular to the planes of the mirrors.



O is the luminous point; A, B are the lines in which the plane of the paper cuts the planes of the mirrors. Let these lines, produced if necessary, meet in C.

Draw OQ_1 perpendicular to A and produce it to Q_1 as far behind A as O is in front of A. Then Q_1 is the image of O formed by reflection at A. Draw Q_1, Q_2 perpendicular to B, and take Q_2 a point as far behind B as Q_1 is in front; Q_2 is the image formed by light reflected first at A and then at B.

In a similar way the positions of the succeeding images can be found, some portion of the light which emanates from any image formed by one mirror being incident on the other mirror.

It is evident that if the position of any of the points Q_1, Q_2...lies within that angle between the mirrors which is vertically opposite to the angle ACB, the light which, being reflected at either mirror, appears to proceed from this image cannot fall on the other mirror; and no more images will be prodused.

It is clear from the construction that $CQ_1 = CO$, similarly that $CQ_2 = CQ_1$, and so on. Thus all the points $Q_1, Q_2 \dots$ lie on a circle whose centre is C and radius CO.

The same will be true of the images formed by successive reflections, if the first image be formed by reflection at B.

Let the angle between the mirrors $ACB = \gamma$,
and let the angle $OCA = \alpha$, $\angle OCB = \beta$,

$$\therefore \alpha + \beta = \gamma.$$

Then

$$\angle OCQ_1 = 2\alpha,$$

$$\angle OCQ_2 = \angle OCB + \angle BCQ_2 = \angle OCB + \angle BCQ_1$$
$$= 2\alpha + 2\beta = 2\gamma,$$

$$\angle OCQ_3 = \angle OCA + \angle ACQ_3 = \angle OCA + \angle ACQ_2 = 2\gamma + 2\alpha,$$

and so on we shall find that generally

$$\angle OCQ_{2n} = 2n\gamma,$$
$$\angle OCQ_{2n+1} = 2n\gamma + 2\alpha.$$

If $P_1, P_2 \dots$ be the images formed by successive reflections beginning with the mirror B, we should similarly arrive at results

$$\angle OCP_{2n} = 2n\gamma,\ OCP_{2n+1} = 2n\gamma + 2\beta.$$

51. We can easily investigate the number of images which will exist for any given values of α, β and γ.

If the last image be an odd one, the point Q_{2n+1} must lie within the angle vertically opposite to ACB, therefore

$$OCQ_{2n+1} > \pi - \beta < \pi + \alpha,$$
$$\therefore 2n\gamma + 2\alpha > \pi - \beta < \pi + \alpha,$$

or adding $\beta - \alpha$ to each side of these inequalities

$$2n\gamma + \alpha + \beta > \pi - \alpha < \pi + \beta,$$
$$2n\gamma + \gamma \quad > \pi - \alpha < \pi + \beta,$$
$$\therefore 2n + 1 > \frac{\pi - \alpha}{\gamma} < \frac{\pi + \beta}{\gamma}.$$

That is $2n+1$ is the integer that lies between $\frac{\pi-a}{\gamma}$ and $\frac{\pi+\beta}{\gamma}$. The difference of these fractions being $\frac{a+\beta}{\gamma}$ or unity, there is only one integer lying between them.

If the last image be an even one we must similarly have

$$OCQ_{2n} > \pi - a < \pi + \beta,$$

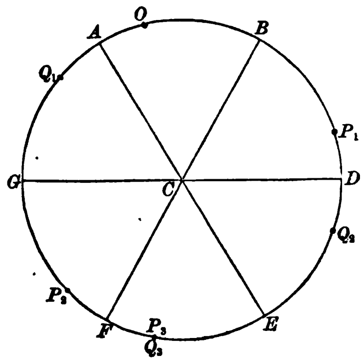
$$\therefore \quad 2n\gamma > \pi - a < \pi + \beta,$$

$$\therefore \quad 2n > \frac{\pi-a}{\gamma} < \frac{\pi+\beta}{\gamma}.$$

In either case the number of images that can be formed is given by the integer that lies between $\frac{\pi-a}{\gamma}$ and $\frac{\pi+\beta}{\gamma}$.

Similarly the number of images that can be formed by reflection first at B can be investigated.

52. The Kaleidoscope furnishes a good illustration of these articles. In that toy two mirrors are inclined at an



angle $\frac{\pi}{3}$ to each other, and bits of coloured glass placed between them at one end of a tube, at the other end of which the eye is placed.

Let CA, CB be the mirrors. Produce BC to F, and AC to E, and draw GCD so as to bisect the angles ACF and BCE.

Then P_1, P_2, P_3 will lie in the angles BCD, GCF and FCE respectively,

Q_1, Q_2, Q_3 in the angles ACG, DCE, ECF respectively.

Also P_3 and Q_3 will coincide; for by the formulæ of Article 50

$\angle OCQ_3$ measured from O in a direction opposite to that in which the hands of a watch move $=\frac{2\pi}{3}+2OCA$, and $\angle OCP_3$ measured in the direction of the motion of the hands of a watch $=\frac{2\pi}{3}+2\angle OCB$.

And the sum of these angles

$$=\frac{4\pi}{3}+2\angle OCA+2\angle OCB$$

$$=\frac{4\pi}{3}+2\angle ACB=\frac{4\pi}{3}+\frac{2\pi}{3}=2\pi.$$

Hence P_3 and Q_3 coincide.

Thus, if there be any arrangement of coloured glass in the compartment ACB, this will be represented over again in the five compartments BCD, DCE, ECF, FCG, GCA, and the eye will see a regular six-fold, or rather three-fold pattern, since the figures will be inverted in alternate compartments.

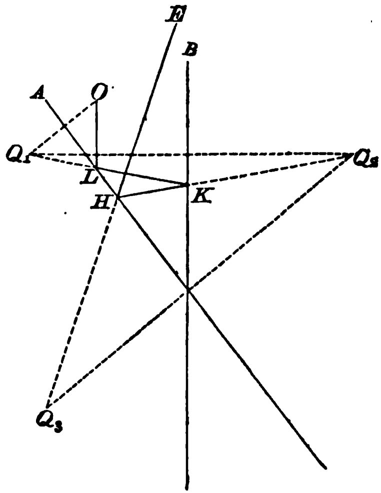
53. To trace the course of the pencil by which an eye placed in any position in the plane of the paper would see any one of the series of images, we must make the following construction.

First, join the image in question with the eye; this joining line obviously must be the direction of the light

when it enters the eye: secondly, join the point where this line cuts the mirror by which the image in question is formed, with the image next before it in order; this line is clearly the direction of the light before the last reflection.

Proceed in this way to determine the direction of the light before each reflection till we finally arrive back at the original source of light.

Thus, if E be the position of the eye, and Q_3 the third image.



Join EQ_3 cutting the mirror A by which Q_3 is formed in H. Join HQ_2 cutting the mirror B in K. Join KQ_1 cutting the mirror A in L. Join LO.

Then a ray of light proceeding from O along OL will be reflected by A along LK, again reflected by B along KH, and finally reflected by A along HE to the eye. These lines will thus indicate the directions of the axes of the oblique pencils by which Q_3 is finally made visible to an eye at E.

EXAMPLES ON CHAPTER IV.

1. Find the angle between two mirrors that a ray reflected at each of them in succession may be moving in a direction at right angles to its first direction.

2. Shew that when an eye is placed to view any image formed by successive reflections at two mirrors, the apparent distance of the image from the eye is equal to the distance actually travelled by the light in coming to the eye from the original point of light.

3. A luminous point is placed at the centre of an equilateral triangle whose side is a, shew that the distance of the image formed by $2n$ reflections at the sides of the triangle in succession from the luminous point is na, and of that formed by $2n+1$ reflections is $a\sqrt{n^2+n+\frac{1}{3}}$.

4. Find the number of images formed when a bright point is placed between two mirrors inclined at an angle of 50^0. Where must the bright point be placed that there may be seven images?

5. A luminous point moves about between two plane mirrors, which are inclined at an angle of 27^0. Prove that at any moment the number of images of the point is 13 or 14 according as the point's angular distance from the nearer mirror is less or not less than 9^0.

6. Find the number of images formed when the angle between the mirrors is 80^0. Find the positions of the point for which there are five images.

7. Two luminous points are placed between two parallel mirrors, on a common perpendicular to their planes: the points are at equal distances from the two mirrors respectively. The distance between the mirrors being a, and between the luminous points c, prove that the distances of the images from each other will be alternately c and $a-c$.

8. A plane and a concave mirror are placed opposite one another on the same axis at a distance apart greater than the radius of the mirror; a person standing with his back to the plane mirror, but close to it, observes the three images of the candle he holds in his hand which are formed by fewest reflections of all that are visible to him. He moves the candle forward

till it coincides with the nearest image: prove that the other two images will coincide also at the same time.

Prove also that if the person moves the candle further forward a distance x till it coincides with another image, at the instant of coincidence the first image will disappear, and if a be the distance between the mirrors, prove that the radius of the concave mirror is

$$\frac{x+a\pm\sqrt{(x+a)^2-8ax}}{2}.$$

9. P is a point within the acute angle AOB formed by two mirrors, and a ray PQR emanating from P is reflected at OA, OB in succession, and returns to P: shew that the length of its path is $2OP \sin AOB$, and that OP bisects the angle QPR.

10. Two small arcs of a circle at the extremities of a diameter are polished and a luminous point is placed in the diameter at a distance u from the centre. Shew that the distance v of the geometrical focus from the centre after m reflections is given by the equation

$$\frac{1}{v}=(-1)^m\left\{\frac{1}{u}\mp\frac{2m}{r}\right\},$$

the upper or lower sign being taken according as the first reflection takes place at the nearer or further arc.

11. Two reflecting paraboloids of revolution are placed with their axes coincident and their concavities turned in opposite directions. Shew that the length of the path of a ray of light in going from any focus and returning to it again is constant. Also find the condition that the ray may retrace its course, and shew that if one ray does so, all the rays will.

12. A ray of light proceeding from a point in the axis of a right cone is incident on the inside of the cone which is polished. After a second reflection the ray retraces its path; shew that the length of its path is $2h \sin 3\alpha$; h being the distance of the origin of light from the vertex, and α the semi-vertical angle of the cone.

13. A luminous point is situated at the centre of the base of a hollow but perfectly reflecting vertical cylinder of very small radius, and a horizontal screen is held over the cylinder at a height above its upper end which is half as great again as the height of the cylinder. Prove that a series of alternately

darker and brighter rings is formed on the screen, the breadths of which are equal to the radius and diameter of the cylinder respectively.

14. An eye looks along the axis of a glass cylinder at the other end of which is a black speck. Prove that the eye will see a number of dark concentric rings, whose centre is the axis of the cylinder; and the number of which is the greatest integer in $\dfrac{h}{a\sqrt{\mu^2-1}}$, where h is the height, a the diameter of the base of the cylinder, and μ the refractive index.

CHAPTER V.

ON REFRACTION THROUGH PRISMS AND PLATES.

54. THE method of determining the course of a single ray of light while passing through a series of media bounded by parallel planes has been fully enough indicated in Articles 14 and 15; we need not therefore consider it farther, but proceed to the determination of the course of a ray of light in passing through a portion of a medium bounded by planes not parallel.

55. A portion of a medium of which two of the boundaries are planes inclined to one another at any angle, is called a prism.

The line of intersection of these planes is called the edge of the prism.

The planes themselves are called the faces of the prism.

The angle between the planes is called the angle of the prism, or sometimes the refracting angle of the prism.

56. When a ray of light passes from one medium to another, the angle between its direction in the first medium and its direction in the second medium is called the *deviation of the ray*.

If a ray of light pass from one medium into another, and ϕ, ϕ' be the angles of incidence and refraction respectively, and μ be the index of refraction between the media, we have

$$\sin \phi = \mu \sin \phi'.$$

From this equation it is evident that as ϕ increases so also does ϕ', and *vice versâ*.

It can farther be proved that as the angle of incidence increases, the deviation increases also.

For $$\sin\phi = \mu\sin\phi',$$

$$\therefore \frac{\sin\phi \sim \sin\phi'}{\sin\phi + \sin\phi'} = \frac{\mu \sim 1}{\mu + 1},$$

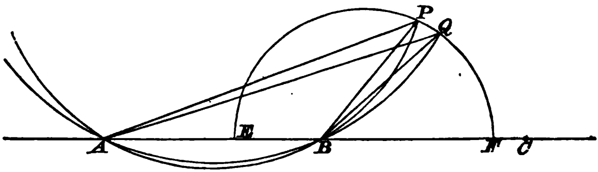
$$\therefore \tan\frac{\phi \sim \phi'}{2} = \frac{\mu \sim 1}{\mu + 1}\tan\frac{\phi + \phi'}{2}.$$

But as ϕ increases, so does ϕ', and therefore $\frac{\phi+\phi'}{2}$; whence $\frac{\phi \sim \phi'}{2}$ increases with ϕ.

But $\phi \sim \phi'$ is evidently the deviation. Hence when a ray of light passes from one medium to another, the deviation increases as the angle of incidence increases.

This proposition can also be proved geometrically.

Let AB be any straight line produced to C.



At A make the angle $BAP = \phi'$, and at B make the angle $CBP = \phi$; BP and AP will meet in some point P, since ϕ and ϕ' are unequal.

The angle BPA is $\phi - \phi'$, that is, the deviation corresponding to the value CBP of ϕ.

Now $$\sin\phi = \mu\sin\phi'.$$

But $$\frac{\sin\phi}{\sin\phi'} = \frac{\sin CBP}{\sin CAP} = \frac{AP}{BP}.$$

Hence $AP = \mu . BP$.

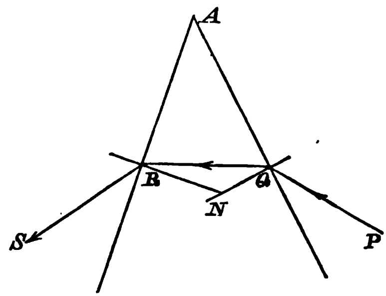
Find a point E between A and B, such that $AE = \mu . BE$, and another point F beyond AB, such that $AF = \mu . BF$.

Then (Euclid, Book VI., Props. 3 and A) EP bisects the interior angle APB of the triangle APB, and FP bisects the exterior angle at P of the same triangle. Hence EPF is a right angle, and the point P must therefore lie on a circle described on EF as diameter.

Take any point Q on this circle, nearer to F than P. Then the angle AQB will be the deviation corresponding to the value QBF of ϕ. But if we describe a segment of a circle through A, B and P, and another through A, B and Q, it is clear that above AB the latter segment must lie entirely outside the former, and the angle in it will be less. Hence the angle AQB is less than the angle APB, that is, the deviation increases with ϕ.

57. When a ray of light passes through a prism of denser material than the surrounding medium, in a plane

Fig. (1).



perpendicular to the edge of the prism, the deviation on the whole is from the edge.

Let the plane in which the light passes be the plane of the paper. Let A be the point in which the edge of the prism meets the plane of the paper, and $PQRS$ be the course of the ray of light.

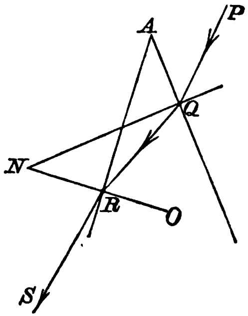
We have three cases.

(1) When the normals at Q and R meet within the prism, as in fig. (1). In this case, since in passing into a

denser medium light is bent nearer to the normal, the deviation at Q is evidently away from the edge, and likewise that at R. Hence, on the whole, the deviation is from the edge.

(2) Let the normals to the faces at Q and R meet without the prism as in fig. (2).

Fig. (2).



The deviation at Q is towards the edge, that at R is from it.

The angle QRO is greater than the angle RQN. Hence, by Article 56, the deviation at R is greater than that at Q.

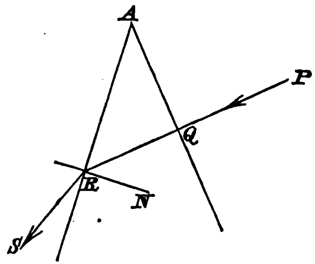
Hence, on the whole, the deviation is from the edge.

The same is the case, if the ray proceeds in the direction $SRQP$, instead of $PQRS$.

(3) Let the normals at Q and R meet on one face of the prism, as at Q in fig. (3).

Then the ray is itself the normal at the point where it meets one face of the prism, and therefore at that point suffers no deviation. At the point where it meets the other face, its deviation is from the edge. Hence, on the whole, the deviation is from the edge.

Fig. (3).



58. It is usual to call the angle of incidence of the ray on the first face ϕ, the corresponding angle of refraction ϕ', the angle of incidence within the prism on the second face ψ', and the anglo which the emergent ray makes with the normal to the second face ψ. We then obviously have, if μ be the refractive index between the external medium and the prism,

$$\sin\phi = \mu \sin\phi',$$
$$\sin\psi = \mu \sin\psi'.$$

59. It is usual and convenient to consider ϕ and ψ positive, if they are measured on the side of the normals to the respective faces away from the edge, and negative, if measured on the side of the normal towards the edge, ϕ' and ψ' are then considered to have the same signs as ϕ and ψ respectively.

With this convention we can show that the algebraic sum of the angles which the ray inside the prism makes with the normals to the two faces is equal to the angle of the prism.

Referring to the figures in Article 57, we see that in figure (1),

$$\phi' = NQR, \psi' = NRQ,$$
$$\therefore \phi' + \psi' = NQR + NRQ$$
$$= 180^\circ - RNQ$$
$$= RAQ.$$

In fig. (2) ϕ is measured towards the vertex, and ϕ and ϕ' are consequently negative;

$$\therefore \; -\phi' = RQN,$$
$$\psi' = QRO,$$
$$\therefore \; \phi' + \psi' = QRO - RQN = RNQ = RAQ,$$

since a circle will go through the four points A, R, Q, N.

In fig. (3) $\qquad \phi' = 0,$

$$\psi' = NRQ = 90^\circ - QRA = RAQ,$$
$$\therefore \; \phi' + \psi' = RAQ.$$

The angle of the prism may be denoted by i; we have therefore always, with the above convention as to signs,

$$\phi' + \psi' = i.$$

60. We can now give a simple expression for the deviation of a ray in passing through a prism.

The deviation of the ray at the first surface is evidently $\phi - \phi'$, from the edge if ϕ is measured from the edge, and towards the edge if ϕ is measured towards the edge.

If we agree to consider a deviation *from the edge* as positive, and to retain the convention of the last Article as to the sign of ϕ, the deviation at the first surface will be algebraically

$$\phi - \phi'.$$

Similarly the deviation at the second refraction will be algebraically

$$\psi - \psi'.$$

Hence the whole deviation will be

$$\phi - \phi' + \psi - \psi'$$
$$= \phi + \psi - i,$$

by the last Article.

If the ray is incident at a small angle on a prism of small angle, so that i and ϕ are small, it follows that ϕ' is small, and therefore ψ', and therefore ψ.

Hence, since the sines of small angles are very nearly equal to their circular measures, we have approximately

$$\phi=\mu\phi', \qquad \psi=\mu\psi',$$

$$\therefore D=\phi+\psi-i=\mu(\phi'+\psi')-i$$

$$=\mu i-i=(\mu-1)i.$$

61. The deviation of a ray in passing through the prism depends on ϕ and ψ. These quantities are connected by the relations

$$\sin\phi=\mu\sin\phi',$$

$$\sin\psi=\mu\sin\psi',$$

where ϕ' and ψ' are connected by the relation

$$\phi'+\psi'=i.$$

We may thus consider D really to be a function of ϕ, for its value will be given for any given value of ϕ, and will change if ϕ be changed. There is one value of ϕ for which the deviation D has a less value than for any other value, and this minimum value of the deviation we proceed to investigate.

62. We know, from Art. 56, that the deviation of a ray in passing from one medium into another increases as the angle of incidence increases.

Let now $\phi'-a$, ϕ', $\phi'+a$ be three angles of refraction corresponding to values ϕ_1, ϕ_2, ϕ_3 of the angle of incidence,

$$\therefore \sin\phi_1=\mu\sin(\phi'-a),$$

$$\sin\phi_2=\mu\sin\phi',$$

$$\sin\phi_3=\mu\sin(\phi'+a),$$

$$\therefore \sin\phi_1+\sin\phi_3=\mu\sin(\phi'+a)+\mu\sin(\phi'-a),$$

$$\therefore 2\sin\frac{\phi_1+\phi_3}{2}\cos\frac{\phi_3-\phi_1}{2}=2\mu\sin\phi'\cos a,$$

$$=2\sin\phi_2\cos a\,\ldots\ldots(1).$$

Now by Art. 56,

$$\phi_3-(\phi'+a)>\phi_1-(\phi'-a),$$

$$\therefore \phi_3-\phi_1>2a,$$

$$\therefore \cos\frac{\phi_3-\phi_1}{2}<\cos\alpha,$$

$$\therefore \sin\frac{\phi_1+\phi_3}{2}>\sin\phi_2 \text{.........} \text{ by (1)},$$

$$\therefore \phi_1+\phi_3>2\phi_2,$$

$$\therefore \phi_3-\phi_2>\phi_2-\phi_1,$$

$$\therefore \{\phi_3-(\phi'+\alpha)\}-(\phi_2-\phi')>(\phi_2-\phi')-\{\phi_1-(\phi'-\alpha)\}.$$

That is, when the angle of refraction increases from ϕ' to $\phi'+\alpha$, the deviation is more increased than when the angle of refraction increases from $\phi'-\alpha$ to ϕ'.

Now let a ray pass through a prism whose refracting angle is i so that $\phi=\psi$; therefore also $\phi'=\psi'=\frac{i}{2}$.

Let ϕ be increased, so that ϕ' is increased and becomes $\frac{i}{2}+\alpha$. It follows that ψ' becomes $\frac{i}{2}-\alpha$.

Hence the deviation is increased at the first surface and diminished at the second, but is more increased at the first than it is diminished at the second. Hence on the whole the deviation is increased.

Similarly if ϕ be diminished, so that ϕ' becomes $\frac{i}{2}-\alpha$, and ψ' consequently becomes $\frac{i}{2}+\alpha$, the deviation is more increased at the second surface than it is diminished at the first.

Hence the least deviation is obtained when $\phi=\psi$, and consequently $\phi'=\psi'=\frac{i}{2}$.

If D_1 be the deviation in this case, we have

$$D_1=2\phi-i,$$

$$\therefore \phi=\frac{D_1+i}{2} \text{ and } \phi'=\frac{i}{2},$$

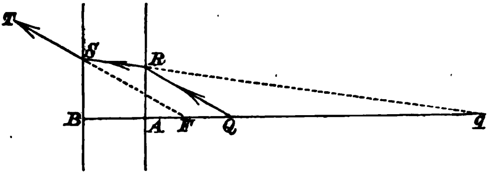
$$\therefore \sin\frac{D_1+i}{2}=\mu\sin\frac{i}{2},$$

which determines D_1.

63. Having discussed the effects of refraction through a plate or a prism on a single ray, we have now to consider the modifications produced by such refractions in a pencil.

We consider, first, the case of a pencil directly incident on a portion of a medium bounded by two parallel planes. Such a portion is frequently called a plate.

Let Q be the origin of the pencil of light, QA the axis of the pencil, which will be also the normal to the first sur-



face at A, and to the second surface at B, since the pencil is *directly* incident.

Let q be the geometrical focus of the pencil after the first refraction into the medium, μ the index of refraction from the external space into this medium.

Then, by Article 24,

$$Aq=\mu . AQ.$$

The pencil after the first refraction may be supposed to be a cone diverging from q, and incident on the external medium. The index of refraction from the plate into this external medium will, by Article 14, be $\frac{1}{\mu}$.

Hence, if F be the geometrical focus after refraction into the external medium again, we have

$$BF=\frac{1}{\mu} . Bq.$$

Let $$AQ=u,\ AB=t,\ AF=v.$$

$$\therefore\ Bq=Aq+t.$$

Thus

$$BF=\frac{1}{\mu}(Aq+t)=\frac{1}{\mu}(\mu\,.\,AQ+t)$$

$$=AQ+\frac{t}{\mu}.$$

And

$$AF=AQ+\frac{t}{\mu}-t$$

$$=AQ-t\,.\,\frac{\mu-1}{\mu},$$

or

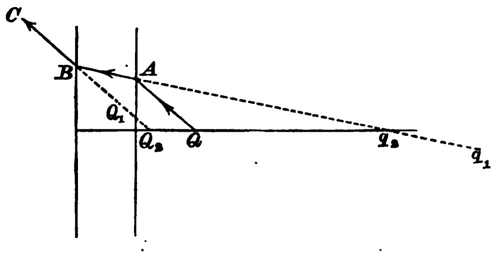
$$v=u-t\,.\,\frac{\mu-1}{\mu}.$$

Hence to an eye on the left of B, the image of the point of light at Q appears nearer than Q by a distance $t\,.\,\frac{\mu-1}{\mu}$.

64. Secondly, let a pencil be obliquely incident on a plate, and pass through it.

In this case, after the first refraction, the pencil does not approximately diverge from a point; and we must separately consider the effect of the refraction on those rays of the pencil which lie in the primary and secondary planes (Arts. 40, 41).

Let QA be the axis of the pencil incident on the plate at A at an angle ϕ, AB the direction of the axis of the



pencil when refracted into the plate making an angle ϕ'

with the normal at A, and BC its direction when finally emergent; BC is evidently parallel to QA.

Let q_1, q_2 be the primary and secondary foci of the pencil after the first refraction.

Then, by Art. 44, if AQ be u, and μ be the index of refraction into the plate,

$$Aq_2 = \mu . u,$$

$$Aq_1 = \mu . \frac{\cos^2\phi'}{\cos^2\phi} . u.$$

After this refraction we may consider those rays which lie in the primary plane to diverge from q_1, and be incident on the second surface of the plate at an angle ϕ', the refractive index being $\frac{1}{\mu}$.

If Q_1 be the focus of the rays in this primary plane after the second refraction, we have again

$$BQ_1 = \frac{1}{\mu} . \frac{\cos^2\phi}{\cos^2\phi'} . Bq_1$$

$$= \frac{1}{\mu} . \frac{\cos^2\phi}{\cos^2\phi'} \left(BA + \mu . \frac{\cos^2\phi'}{\cos^2\phi} u \right).$$

But if t be the thickness of the plate,

$$AB = \frac{t}{\cos\phi'},$$

$$\text{whence } BQ_1 = \frac{t}{\mu} . \frac{\cos^2\phi}{\cos^3\phi'} + u \text{ .... (1).}$$

Again, after the first refraction, we may consider the rays which lie in the secondary plane, to diverge from q_2 and fall on the second surface, and be there refracted again, the refractive index being $\frac{1}{\mu}$.

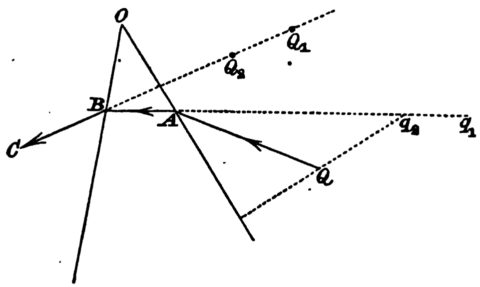
Hence we have, if Q_2 be the focus of the rays in the secondary plane after this refraction,

$$BQ_2 = \frac{1}{\mu} . Bq_2$$
$$= \frac{1}{\mu}(AB + Aq_2)$$
$$= \frac{t}{\mu \cos \phi'} + u.$$

Those rays of the emergent pencil which lie in the primary plane, approximately diverge from a point Q_1; and the rays in its secondary plane approximately diverge from a point Q_2.

It is easy to see that the general form of such a pencil must nearly be the same as that of a pencil after one oblique refraction, as described in Art. 42, and that there will be a circle of least confusion determined, as in that Article, from the positions of Q_1 and Q_2.

65. We have finally to determine the form of a pencil after oblique refraction through a prism.



Let QA be the axis of such a pencil, incident on the first face of a prism, refracted along AB, and finally emergent along BC.

Let ϕ, ψ, ϕ', ψ' have the meanings assigned to them in Article 58.

Also let q_1, q_2 be the foci of the rays in the primary and secondary planes respectively after the first refraction, Q_1, Q_2 the corresponding foci after the second refraction.

Let $AB=c$, $AQ=u$.

Then, by Art. 44, for the first refraction, we have

$$Aq_1=\mu\,.\,\frac{\cos^2\phi'}{\cos^2\phi}\,.\,u,$$

$$Aq_2=\mu u.$$

And from the second refraction, remembering that the index of refraction is $\frac{1}{\mu}$, we have

$$BQ_1=\frac{1}{\mu}\,.\,\frac{\cos^2\psi}{\cos^2\psi'}\,.\,Bq_1,$$

$$BQ_2=\frac{1}{\mu}\,.\,Bq_2,$$

$$\therefore\ BQ_1=u\,.\,\frac{\cos^2\psi\,.\,\cos^2\phi'}{\cos^2\psi'\,.\,\cos^2\phi}+\frac{c}{\mu}\,.\,\frac{\cos^2\psi}{\cos^2\psi'},$$

$$BQ_2=u+\frac{c}{\mu}.$$

It is usual to consider the axis of the pencil to pass so near to the edge of the prism, or else the size of the prism to be so small, that AB may be neglected in comparison with the other distances. This will generally be a tolerably accurate supposition in practical work. We then get

$$BQ_1=u\,.\,\frac{\cos^2\psi\,.\,\cos^2\phi'}{\cos^2\psi'\,.\,\cos^2\phi},$$

$$BQ_2=u.$$

These equations give the foci of the rays in the primary and secondary planes of the emergent pencil respectively. The circle of least confusion can be deduced, as in Art. 42, in accordance with the remark at the end of Art. 64.

If the axis of the pencil be incident so as to pass through the prism with minimum deviation,

$$\phi=\psi \text{ and } \phi'=\psi',$$

we then get

$$BQ_1 = u,\ BQ_2 = u,$$

that is, the rays in the primary and the secondary planes diverge approximately from the same point. Thus the whole pencil may be considered in this case approximately to diverge from one point at a distance from the edge of the prism equal to that of the original point of light.

EXAMPLES ON CHAPTER V.

1. Shew that if the angle of a prism be greater than twice the critical angle for the medium of which it is composed, no ray can pass through.

2. A ray of light is incident upon one face of a prism, in a direction perpendicular to the opposite face. Shew that, i the angle of the prism being less than 90^0, the ray will emerge at the opposite face if $\cot i > \cot \alpha - 1$, where α is the critical angle for the medium of which the prism is composed.

3. If a ray be incident nearly parallel to the first surface of a prism and emerge at right angles to that surface; prove that $\cot i + 1 = \sqrt{\mu^2 - 1}$, i being the angle of the prism, and μ the refractive index.

4. Rays are incident at a given point of a prism so as to be refracted in a plane perpendicular to its edge. If i be the angle of the prism and α the critical angle, show that the angular space within which rays may be incident so as to pass through the prism is $\cos^{-1}\left\{\dfrac{\sin(i-\alpha)}{\sin \alpha}\right\}$.

5. If the angle of a prism be 60^0 and the refractive index $\sqrt{2}$, shew that the minimum deviation is 30^0.

6. The angle of a prism is 60^0 and the refractive index $\frac{3}{2}$. Shew that the minimum deviation of a ray of light passing through it is nearly $37^0\ 10'$; having given that $\sin 48^0\ 35' = {\cdot}75$ nearly.

7. If D be the minimum deviation for a prism, whose refractive index is μ and angle i, prove that

$$\cot\frac{i}{2}+\cot\frac{D}{2}=\mu\,\text{cosec}\,\frac{D}{2}.$$

8. If D_1 be the minimum deviation for a prism of angle i, and D_2 that for a prism of the same material of angle $2i$, prove that

$$2\cos\left(i+\frac{D_1+D_2}{4}\right).\sin\frac{D_2-D_1}{4}=\sin\frac{D_1}{2}.$$

9. A ray passes through n equal prisms, in each case with minimum deviation. If its final direction is parallel to its direction at incidence, and it be moving towards the same part, prove that with the usual notation

$$\tan\phi=\frac{\mu\sin\frac{\pi}{n}}{\mu\cos\frac{\pi}{n}-1},\quad i=2\tan^{-1}\frac{\sin\frac{\pi}{n}}{\mu-\cos\frac{\pi}{n}}.$$

10. If the angle of a prism be 60^0, and the refractive index $\sqrt{\frac{7}{3}}$, find the limits between which ϕ must lie that the ray may be able to emerge at the second face.

11. If ϕ be the angle of incidence on a prism, ψ that of emergence, i the refracting angle of the prism, and μ the refractive index, prove that

$$\mu^2\sin^2 i=(\sin\psi+\cos i\sin\phi)^2+\sin^2 i\,.\,\sin^2\phi.$$

12. A small pencil of rays is refracted through a prism in a principal plane. Shew that if the emergent pencil diverge from a point,

$$u=\frac{c}{\mu}.\frac{\tan^2\psi}{\tan^2\phi-\tan^2\psi},$$

u, ϕ, ψ, c having the meanings given to them in Art. 65.

13. The minimum deviation for a prism is 90^0. Shew that the least value possible for the refractive index is $\sqrt{2}$.

14. If the minimum deviation for rays incident on a prism be α, the refractive index cannot be less than $\sec\frac{\alpha}{2}$, and the angle of the prism cannot be greater than $\pi-\alpha$.

15. Two parallel rays are incident on one face of an isosceles prism at an angle ϕ, and emerge at right angles, one of them

having been reflected at the base. If i be the angle of the prism, and μ its refractive index, prove that

$$\sin 2\phi = (1 - \mu^2 \sin^2 i) \sec i.$$

16. Shew that when a prism of glass of small refracting angle is immersed in water, the deviation of a ray passing through it is only one-fourth of what it is in air.

17. Shew that when a ray of light enters a medium whose refractive index is $\sqrt{2}$, its greatest deviation is 45°.

18. A small pencil of light is obliquely refracted through a plate of thickness t. The angle of incidence being $\tan^{-1}\mu$, show that the distance between the secondary focus after emergence, and the original point of light is $\frac{\mu^2 - 1}{\mu^2} . t$.

19. The angles at the base of a triangular prism are $\theta - \phi$ and 2θ, where $\sin \theta = \mu \sin \phi$; a ray of light falls on the shorter side of the triangle, the angle of incidence being θ on the side of the normal next the vertex: show that the ray after reflection from the base and the other side will emerge from the base in a direction parallel to its original direction; and that unless $\sin^2\theta$ is greater than $\sin \phi$, the second reflection will not be total.

20. A ray of light is refracted through a sphere of glass in such a manner that it passes through the extremities of two radii at right angles to each other. If ϕ be the angle of incidence, and D the deviation, prove that

$$\sin (2\phi - D) . \sin D = \mu^2 - 1.$$

21. If n equal and uniform prisms be placed on their ends with their edges outwards, find the angle of each prism that a ray refracted through each of them in a plane perpendicular to their edges may describe a regular polygon. Shew that the distance of the point of incidence of such a ray on each prism from the edge of the prism bears to the distance of each edge from the common centre the ratio of $\sqrt{\left(\mu^2 - 2\mu \cos \frac{\pi}{n} + 1\right)}$ to $\mu + 1$.

22. Prove that in prisms of the same material, as the refracting angle increases the minimum deviation also increases.

CHAPTER VI.

ON REFRACTION THROUGH LENSES.

66. A PORTION of a transparent medium bounded by surfaces, two of which are surfaces of revolution with a common axis, is called a lens.

In all cases that we shall have to consider, these surfaces of revolution are spheres, and the portion of the medium is symmetrical with respect to the line joining the centres of the spheres, being either entirely bounded by the surfaces of the spheres, or by them and a cylindrical surface, whose axis is the line joining the centres of the spheres.

The spherical surfaces are called the faces of the lens.

The line joining the centres of the spheres is called the axis of the lens.

Lenses of different forms are distinguished by names indicating the nature of their bounding surfaces with respect to the external medium.

A lens, of which both spherical boundaries are convex towards the outside, is called a double convex lens.

A lens, of which one face is convex, and the other concave to the outside, is called a convexo-concave, or concavo-convex lens, according as the light falls first on the convex or concave face respectively.

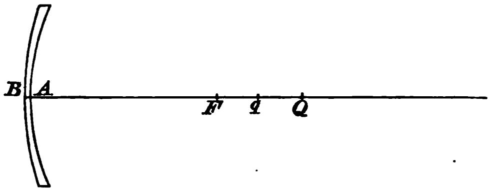
A lens, of which one face is convex, and the other plane, is called convexo-plane or plano-convex.

The terms double concave, concavo-plane, and plano-concave, are intelligible without farther explanation.

A lens, which is on the whole thicker at the middle than at the borders, is called generally a convex lens, while one, which is on the whole thinner at the middle than at the borders, is called a concave lens.

A concavo-convex or convexo-concave lens, which is thicker in the middle than at the borders, is sometimes called a meniscus.

67. A pencil is incident on a lens of small thickness in such a manner that its axis before refraction coincides with



the axis of the lens. It is required to find its geometrical focus after refraction through the lens.

It is clear that the axis of the pencil, after refraction into the lens, and again after emergence, will still coincide with the axis of the lens.

Let Q be the origin of light, QAB the common axis of the pencil and the lens. Let r, s be the radii of the first and second surfaces of the lens respectively, μ the refractive index from the external medium into the lens.

Let q be the geometrical focus of the pencil after the first refraction, F its geometrical focus after emergence. Let AB, the thickness of the lens, be so small that we may neglect it in comparison with AQ and AF.

Let $$AQ=u, \; AF=v.$$

Then for the first refraction, by Article 29, we have

$$\frac{\mu}{Aq}-\frac{1}{u}=\frac{\mu-1}{r}.$$

For the second refraction we may consider the pencil to diverge from q, and F to be its geometrical focus, and remembering that the index of refraction from the lens into the external medium is $\frac{1}{\mu}$, we have

$$\frac{\frac{1}{\mu}}{AF}-\frac{1}{Aq}=\frac{\frac{1}{\mu}-1}{s},$$

or

$$\frac{1}{v}-\frac{\mu}{Aq}=-\frac{\mu-1}{s},$$

adding this to the former equation we have

$$\frac{1}{v}-\frac{1}{u}=(\mu-1)\left(\frac{1}{r}-\frac{1}{s}\right)\dots\dots\dots\dots(1).$$

If the original pencil consist of parallel rays, u is infinite, and if the corresponding value of v be f, we have

$$\frac{1}{f}=(\mu-1)\left(\frac{1}{r}-\frac{1}{s}\right)\dots\dots\dots\dots(2).$$

This quantity f is called the *focal length* of the lens, and the geometrical focus of a pencil of parallel rays incident on the lens parallel to the axis, is called the principal focus of the lens.

The points Q and F are called conjugate foci. By means of formula (2) the relation (1) can be written

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f}\dots\dots\dots\dots(3).$$

If the thickness AB be not neglected, and u be measured from A, and v from B, the second equation becomes

$$\frac{1}{v}-\frac{\mu}{Bq}=-\frac{\mu-1}{s},$$

whence

$$\frac{Bq}{\mu}=\frac{1}{\frac{1}{v}+\frac{\mu-1}{s}},$$

also

$$\frac{Aq}{\mu} = \frac{1}{\frac{1}{u} + \frac{\mu - 1}{r}}.$$

Hence if $AB = t$, we obtain

$$\frac{Bq - Aq}{\mu} = \frac{t}{\mu} = \frac{1}{\frac{1}{v} + \frac{\mu - 1}{s}} - \frac{1}{\frac{1}{u} + \frac{\mu - 1}{r}} \ldots\ldots (4).$$

The formulæ (1) or (3) are however sufficiently accurate in almost all cases, and will be used hereafter.

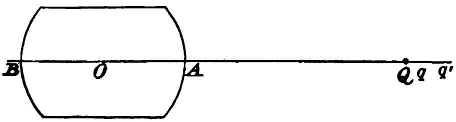
68. It is easy to see that if r and s have opposite signs, the lens is double concave or double convex. In the former case r is positive and s negative and therefore f is positive. Similarly in the latter case f is negative.

If r and s be both positive, the lens is concavo-convex, and f will be positive or negative according as $\frac{1}{r} >$ or $< \frac{1}{s}$, that is, as the *curvature* of the first face is greater or less than that of the second, that is, according as the lens is on the whole concave or convex.

If r and s be both negative, the lens is convexo-concave, and it will again appear that f is positive or negative according as the lens is on the whole concave, or on the whole convex.

Thus we can say generally that the focal length of a concave lens is positive, and the focal length of a convex lens is negative.

69. There is one case of a lens, having a thickness which may be called considerable, that possesses some practical interest, namely, when the bounding surfaces are portions of the same sphere.



In this case it is convenient to use the formula of Art. 37, so as to have the same point of reference for the two refractions.

Let O be the centre of the sphere of which the faces of the lens are portions; let Q be the origin of light and q' and q the foci after refraction at the first and second surfaces respectively. Let $OQ=p$, $Oq=q$, and $OA=r$.

For the first refraction, we have

$$\frac{\mu}{p}-\frac{1}{Oq'}=\frac{\mu-1}{r}.$$

For the second refraction, remembering that the refractive index is $\frac{1}{\mu}$, and the radius OB, being drawn to the left from O, must be considered negative, we have

$$\frac{\frac{1}{\mu}}{Oq'}-\frac{1}{q}=-\frac{\frac{1}{\mu}-1}{r}.$$

$$\therefore \quad \frac{1}{Oq'}-\frac{\mu}{q}=\frac{\mu-1}{r}.$$

$\therefore$ adding these equations, we have

$$\frac{\mu}{p}-\frac{\mu}{q}=\frac{2(\mu-1)}{r},$$

$$\therefore \quad \frac{1}{q}-\frac{1}{p}=-\frac{2(\mu-1)}{\mu r}.$$

A lens of the above form has the advantage that any line through O may be considered as its axis, and thus pencils from any point may be considered as directly incident. Such lenses have accordingly been sometimes used as simple magnifiers instead of the ordinary lenses (Art. 85).

70. The changes of relative position of the conjugate foci of a lens can be traced out in a similar manner to that adopted in Article 34.

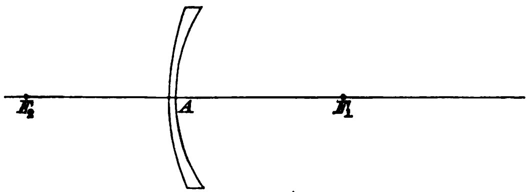
We will take the case of a concave lens, when consequently f is positive.

From the formula

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f},$$

we see that as u increases, so must v; and if u decreases, so must v. Hence the conjugate foci move in the same direction.

Take a point F_1 in the axis of the lens, in front of the lens at a distance f from A, and also a point F_2 behind the lens at an equal distance from A.



When Q is at an infinite distance to the right of A, the incident rays are parallel, and u is infinite; we thus get $v=f$, or F is at F_1.

As Q travels up towards A, F also travels towards A, and when u is very small, $\frac{1}{u}$ being very large, it follows that $\frac{1}{v}$ is very large, and therefore v is very small. Hence when Q gets to A, F also arrives at A.

When Q passes to the left of A so that u is negative, v will be negative until u is numerically equal to f, which is the case when Q is at F_2. Hence while Q travels from A to F_2, F travels from A to an infinite distance to the left.

When Q is at F_2, $u=-f$, and therefore $v=\infty$. Hence the refracted pencil consists of parallel rays.

As Q travels to the left of F_2, u is negative and numerically greater than f. Hence v is positive and F

travels up from a great distance to the right, towards F_1; which point it reaches when Q has gone to an infinite distance to the left and the incident rays are again parallel.

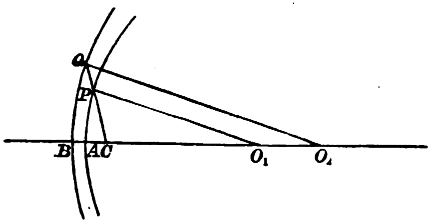
The student can exercise himself in tracing the changes of position of Q and F in the case of a convex lens.

71. Before discussing the form of an oblique pencil after refraction through a lens, we must investigate the position of a point which is known as *the centre of the lens.*

The centre of a lens is the point in which lines joining the extremities of parallel radii of the two bounding surfaces cut the axis.

This point is one of the centres of similitude of the two spherical surfaces; we proceed to find its position.

Let BAO_1O_2 be the axis of the lens, O_1P any radius of the first surface; O_2Q a parallel radius of the second surface.



Let QP or QP produced meet BAO_1O_2 in C. Then C is the centre of the lens and is a fixed point whatever pair of parallel radii we employ.

Let AB the thickness of the lens $=t$, $AO_1=r$, $BO_2=s$. Then by similar triangles

$$CO_1 : CO_2 :: O_1P : O_2Q$$
$$:: r : s,$$

$$\therefore \frac{r-AC}{s-t-AC}=\frac{r}{s}; \quad \therefore AC=\frac{rt}{s-r}.$$

If t be very small, AC is very small and the centre nearly coincides with A or B.

The centre of a lens, determined as above, has the following important optical property.

Any ray passing through a lens in such a manner that its direction, while within the lens, passes through the centre, will on emerging from the lens have a direction parallel to its direction when incident on the lens.

This follows at once from the fact that the normals at the two points where refraction in this case takes place are parallel, and therefore the effect on this ray is the same as if it had been refracted through a plate.

72. We have now to distinguish between the cases of oblique refraction through a lens.

First, *centrical refraction*, when the central ray or axis of the pencil passes through the centre of the lens after refraction at the first surface.

In this case the axis of the pencil undergoes no deviation. Refraction through a lens is usually centrical when light from any natural object falls upon the lens so as to fill up its whole surface. It may happen however, as in Galileo's telescope (Art. 98), that only a portion of this light is utilised afterwards for purposes of vision, in which case we have an example of the second kind of oblique refraction or

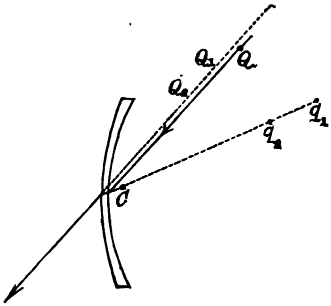
Excentrical refraction: A pencil is said to be excentrically refracted through a lens when the axis of the pencil, while within the lens, does not pass through the centre of the lens.

The axis of such a pencil does therefore undergo deviation in passing through the lens.

Excentrical refraction usually takes place when light from an image formed by reflection or refraction falls on a lens. The light emitted by such an image differs from that emitted by a real object in that it only can diverge from the image in the lines in which it had previously converged to form the image. Hence the pencil of light

from any point of such an image is limited by its own nature, and not by the lens on which it falls.

73. We can now discuss the form of a pencil obliquely and centrically refracted through a thin lens.



Let Q be the origin of light, and let q_1, q_2 be the primary and secondary foci after the first refraction. Let Q_1, Q_2 be the primary and secondary foci after the second refraction.

Let ϕ be the angle of incidence of the axis at the first surface, which is also the angle of emergence at the second.

Let us suppose the lens so thin that A, B and C may sensibly coincide, and let μ be the index of refraction into the lens;

that $$CQ = u,\ CQ_1 = v_1,\ CQ_2 = v_2.$$

Then for the first refraction, by Art. 46,

$$\frac{\mu \cos^2\phi'}{Cq_1} - \frac{\cos^2\phi}{u} = \frac{\mu \cos\phi' - \cos\phi}{r},$$

$$\frac{\mu}{Cq_2} - \frac{1}{u} = \frac{\mu \cos\phi' - \cos\phi}{r}.$$

Again for the second refraction, remembering that the

index of refraction is $\frac{1}{\mu}$, and that ϕ' is the angle of incidence, and ϕ that of refraction, we have

$$\frac{\frac{1}{\mu}.\cos^2\phi}{v_1} - \frac{\cos^2\phi'}{Cq_1} = \frac{\frac{1}{\mu}\cos\phi - \cos\phi'}{s},$$

$$\frac{\frac{1}{\mu}}{v_2} - \frac{1}{Cq_2} = \frac{\frac{1}{\mu}\cos\phi - \cos\phi'}{s}.$$

Multiplying these equations by μ, and adding to the former pair, each to each, we have

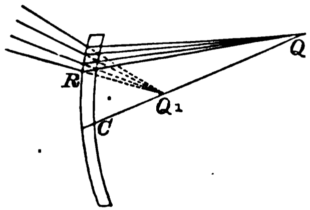
$$\frac{\cos^2\phi}{v_1} - \frac{\cos^2\phi}{u} = (\mu\cos\phi' - \cos\phi)\left(\frac{1}{r} - \frac{1}{s}\right),$$

$$\frac{1}{v_2} - \frac{1}{u} = (\mu\cos\phi' - \cos\phi)\left(\frac{1}{r} - \frac{1}{s}\right).$$

If ϕ be small, which is usually the case in practice, $\cos\phi$ and $\cos\phi'$ may be taken as unity and we obtain $v_1 = v_2$. Each of these quantities has the same value as that of v in the formula for a direct pencil, namely

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}.$$

74. The exact investigation of the form of a pencil after excentrical refraction through a lens is much too difficult for an elementary treatise. It will be sufficient to consider any such pencils as small portions of centrical oblique pencils.



Thus if QR be the axis of a pencil excentrically incident on a lens at R; we may consider this pencil as a portion of a large oblique pencil, whose axis is QC, C being the centre of the lens.

If Q_1 be the focus of this larger pencil, determined in accordance with the remark at the end of the last Article, Q_1 will lie in CQ and will be approximately the focus of the smaller pencil after refraction.

Thus we shall assume that when a small pencil is excentrically refracted through a lens, after refraction it diverges from, or converges to, some point in the line joining the origin of light with the centre of the lens.

75. We have thus far considered refraction through one lens only. There are two cases in which it is important to consider refraction through two or more lenses.

The first case is that of centrical refraction through two thin lenses having the same axis and placed in contact with each other at their centres.

Let a direct pencil be incident centrically on such a combination. Let f_1 be the focal length of the first lens, f_2 that of the second.

We shall suppose the lenses so thin that their centres may be supposed coincident. In almost all cases that occur, the thickness of the lenses is a very small quantity compared with the other lengths that are involved.

Let u be the distance of the origin of light from the common centre of the lenses, v' the distance from the same point of the geometrical focus after refraction through the first lens, and v the distance of the final geometrical focus from the same point.

Then we have by Art. 67, equation (3),

$$\frac{1}{v'} - \frac{1}{u} = \frac{1}{f_1},$$

$$\frac{1}{v} - \frac{1}{v'} = \frac{1}{f_2},$$

∴ adding

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f_1}+\frac{1}{f_2}.$$

If F be the focal length of a single lens which, if placed with its centre at the same point as the common centre of the above lenses, would produce the same refraction in the pencil, we have

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{F},$$

therefore

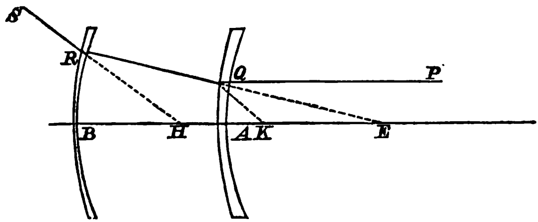
$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}.$$

The lens whose focal length is F is sometimes said to be *equivalent* to the combination of the two lenses.

A similar formula will easily be seen to hold for the focal length of a lens equivalent to any number of lenses with a common axis, placed in contact.

76. The term *equivalent lens* is more usually defined in the following manner.

One lens is said to be equivalent to a combination of two or more lenses having a common axis, when it produces the same deviation in the axis of an excentrical pencil incident parallel to the axis, as the combination does, the equivalent lens being placed in the position of that lens on which the light falls first.



Let f_1, f_2 be the focal lengths of the lenses, PQ the axis of a pencil incident excentrically at Q on the first lens, PQ

being parallel to the common axis of the lenses. Let QR, RS be the directions of this axis after refraction through the first and second lenses respectively, meeting the axis of the lenses in E and H respectively. Let A, B be the centres of the two lenses and let $AB = a$.

Then, PQ being parallel to the axis of the lens A, we have $AE = f_1$.

Also, since we may consider QR to be a ray of a pencil proceeding from E and incident on the second lens, we have

$$\frac{1}{BH} - \frac{1}{BE} = \frac{1}{f_2},$$

$$\frac{1}{BH} = \frac{1}{f_1 + a} + \frac{1}{f_2}.$$

$$\therefore \quad BH = \frac{f_2(f_1 + a)}{f_1 + f_2 + a} \quad \text{.....................(1).}$$

Again if F be the focal length of the required equivalent lens, and we draw QK parallel to HRS to meet BA in K, it is clear that $AK = F$, since the equivalent lens, placed with its centre at A, is to bend the ray PQ so as to be parallel to RS.

Also if we suppose the curvatures of the lenses small, we have by similar triangles

$$\frac{AQ}{BR} = \frac{AE}{BE} = \frac{f_1}{f_1 + a}.$$

Also
$$\frac{AQ}{BR} = \frac{AK}{BH};$$

$$\therefore \quad AK = F = \frac{BH . f_1}{f_1 + a} = \frac{f_1 f_2}{f_1 + f_2 + a} \quad \text{...... by (1),}$$

$$\therefore \quad \frac{1}{F} = \frac{f_1 + f_2 + a}{f_1 f_2} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{a}{f_1 f_2} \quad \text{.......... (2).}$$

We have drawn a figure in which both lenses are concave and consequently F, f_1 and f_2 are all positive. In the most important cases in practice the reverse is the case, f_1 and f_2 and consequently F being negative.

The formula giving the numerical value of F in terms of the numerical values of the focal lengths of the two lenses is deduced from (2) by altering the signs of F, f_1, f_2.

We thus have,

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{a}{f_1 f_2} \text{..................... (3).}$$

Either of these formulæ includes that of the last Article, which may be deduced by making a to vanish.

Any other case of refraction through a number of lenses may be similarly treated; by considering the point to which the pencil converges, or from which it diverges after refraction through one lens, as the origin of the pencil incident on the next lens, the geometrical focus after refraction through any number of lenses can be determined.

EXAMPLES. CHAPTER VI.

1. Prove that the centre of a lens, of which one bounding surface is a plane, lies on the curved surface. Also that the centre of a double concave or double convex lens lies within the lens.

2. A concave mirror, of radius r, has its centre at the centre of a convex lens, and the axes of the two coincide. If f be the focal length of the lens, and if rays proceeding from a point at a distance u from the lens, after refraction through the lens, reflection at the mirror, and a second refraction through the lens, emerge as a pencil of parallel rays, prove that

$$\frac{1}{u} + \frac{2}{r} = \frac{2}{f}.$$

3. A sphere of glass, of radius R, has a concentric spherical cavity of radius r. A pencil of parallel rays is directly refracted through the shell. Shew that the distance of the geometrical focus from the centre of the spheres after emergence is

$$\frac{1}{2}\frac{\mu}{\mu-1} \cdot \frac{Rr}{R-r},$$

-ing the index of refraction.

4. A hemisphere of glass has its spherical surface silvered; light is incident from a luminous point Q, in the axis of figure produced, on the plane surface. Shew that if q is the geometrical focus of the pencil after refraction into the hemisphere, reflection at the silvered surface and again refraction out of the hemisphere,

$$\frac{1}{Aq} - \frac{1}{AQ} = -\frac{2\mu}{OA},$$

A being the centre of the hemisphere, O its vertex, and μ the refractive index of glass.

5. The ends of a glass cylinder are worked into convex spherical surfaces whose radii are equal to the length of the cylinder, and whose centres are at the ends of the axis of the cylinder. Prove that the geometrical focus of a pencil after direct refraction through the ends of the cylinder is determined by the equation

$$\frac{\mu^2}{v} - \frac{1}{u} = -\frac{\mu^2 - 1}{r},$$

where u and v are measured from the face nearest the origin of light, and r is the length of the cylinder.

6. A ray of light is incident on a portion of a refracting medium in the shape of a prolate spheroid, parallel to the axis. The excentricity of the generating ellipse of the spheroid being $\frac{1}{\mu}$, shew that the deviation of the ray after emergence at the opposite side will be twice the angle which the normal at the point of emergence makes with the axis.

7. A solid transparent sphere is composed of a small solid sphere of radius a, and two concentric spherical shells each of thickness a. The refractive indices of these beginning from the centre are 4μ, 2μ, μ, respectively. A pencil of rays is incident directly on the sphere and, after refraction through all three substances, on emerging diverges from a point on the outer surface. Shew that the incident pencil converges to a point whose distance from the centre of the sphere is $\frac{3\mu a}{\mu + 1}$.

8. Shew that the focal length of the sphere formed by two equal hemispheres of glass of different kinds, is equal to the focal length of an equal sphere of glass whose refractive index is

a harmonic mean between the refractive indices of the two hemispheres.

9. From a cubic inch of glass ($\mu = \frac{3}{2}$) the inscribed sphere is removed, a film of glass remaining at the points of contact. The cavity is filled with water ($\mu = \frac{4}{3}$). A bright point is placed on the axis at a distance of one inch from one face of the cube. Find the geometrical focus after refraction through the cube.

10. A hollow spherical shell of glass ($\mu = \frac{3}{2}$) is filled with water ($\mu = \frac{4}{3}$). Shew that a pencil of parallel rays after passing through the whole will converge to a point at a distance from the surface of the glass equal to $(r+t)\dfrac{3r+t}{3r-t}$, where r is the radius of the water sphere, and t the thickness of the glass.

11. The front surface of a mirror is spherical, the back, which is silvered, is plane. If α, β be the distances from the centre of the mirror of the two images of a luminous point placed in the axis of the mirror, which are formed by reflection at the back and front of the mirror respectively, shew that

$$\frac{1}{\alpha} + \frac{1}{\beta} = \text{constant},$$

the thickness of the mirror being neglected.

12. A transparent sphere, radius a, is silvered at the back, and there is a speck within it, half way between the centre and the silvered side. Prove that the distance between the images formed, (1) by one refraction, (2) by one reflection and one refraction, is

$$\frac{2\mu a}{(3-\mu)(\mu-1)}.$$

13. A lens is placed at the centre of a concave mirror, the axes being coincident; a pencil is incident directly on the lens, and after refraction is reflected at the mirror and again refracted through the lens: prove that the last geometrical focus is the same as if the pencil had been once reflected at a mirror coincident with the image of the concave mirror formed by the lens.

14. The focal length of a double equiconcave lens, whose refractive index is $\frac{3}{2}$, is five inches; prove that the distances from the lens of the images of a distant object formed (1) by reflection at the first surface, (2) by one reflection at the second surface, (3) by two reflections at the second surface, are $2\frac{1}{2}$ inches, $1\frac{1}{4}$ inch, and $\frac{1}{2}$ inch respectively.

15. Prove that if a double convex lens be constructed with each of its surfaces a hyperboloid of revolution with excentricity equal to the refractive index, a pencil of rays diverging from the external focus of one surface will be *accurately* refracted to the external focus of the other.

16. An object O is placed in front of two lenses P and Q having a common axis, and an image of it is formed by them: prove that the position of that image will not be altered by interposing between P and Q two lenses of equal and opposite focal lengths, provided that the absolute focal length of either be half the harmonic mean between their distances from the image of O formed by P.

17. Two lenses whose focal lengths are each equal to f are placed at a distance apart equal to $\frac{2}{3}f$. Find the focal length of the equivalent lens.

18. Two lenses whose focal lengths are f and $3f$ are placed at a distance apart equal to the difference of their focal lengths. Find the focal length of the equivalent lens.

19. Two lenses, one concave and the other convex, are placed in contact and have a common axis. Their focal lengths are required to be in the ratio of 52 to 33, and the focal length of the combination is to be six feet. Find the focal length of each lens.

20. Two lenses whose focal lengths are each f are placed at a distance $\frac{1}{2}f$ apart. Find the focal length of the equivalent lens. What is the focal length of the equivalent lens, when the two lenses are placed in contact?

21. The radii of the surfaces of a lens are $4a$ and $2a$. Those of the surfaces of another $2a$ and $6a$. Find the focal length of the lens which is equivalent to them when placed in contact, the refractive index being $\frac{3}{2}$.

22. A small pencil of light is obliquely incident on a refracting sphere, and emerges after one internal reflection. Find the positions of the primary and secondary foci after emergence. Find what the angle of incidence must be that the emergent rays in the primary plane may be parallel.

23. A ray falls upon a lens making the incident angle equal to $\tan^{-1}\mu$, μ being the index of refraction. Its direction within

the lens passes through the centre. Prove that the distance between its directions before incidence, and after emergence, is to the distance it traverses within the lens as $\mu^2 - 1$ to $\mu^2 + 1$.

24. The focal length of a lens in vacuo is five feet. The refractive indices of glass and water being $\frac{3}{2}$ and $\frac{4}{3}$ respectively, find the focal length of the lens when placed in water.

25. A pencil of parallel rays is refracted through a sphere of radius r and refractive index μ. Prove that the geometrical focus after emergence is at the same point as if it had been only refracted at the first surface of a concentric sphere of radius $\frac{\mu r}{2}$.

26. Two convex lenses, of focal lengths a and b, are placed at a distance c: if P and Q be conjugate foci, F, G the respective positions of P and Q when Q and P are respectively at an infinite distance, prove that $PF \,.\, GQ = \left(\frac{ab}{a+b-c}\right)^2$.

CHAPTER VII.

ON IMAGES AND SIMPLE OPTICAL INSTRUMENTS.

77. **WE can now explain the manner in which an image or representation of an object is produced by a lens.**

We will first take the case of a convex lens.

Let *C* be the centre of such a lens, *CQ* its axis, *PQ* the object.

Fig. (1).

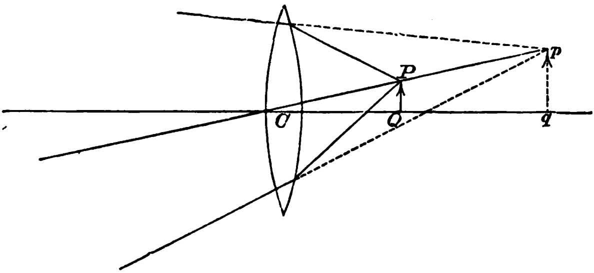
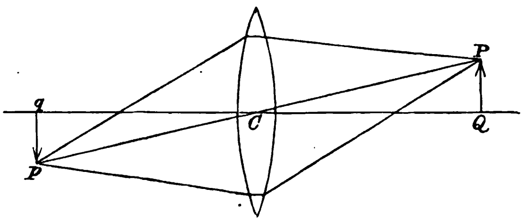


Fig. (2).



Then a pencil of rays from any point P in this object will fall upon the lens so as to cover the whole face of the lens, and will thus be incident centrically. This pencil after refraction will approximately converge to a point p in PC produced, or diverge from some point p in CP produced, the distance Cp being given by the formula

$$\frac{1}{Cp} = \frac{1}{CP} - \frac{1}{f} \quad \text{(Art. 73, end)},$$

f being the numerical value of the focal length of the lens. If $CP < f$, Cp is positive, and p lies to the right of C. If $CP > f$, Cp is negative and p lies in PC produced.

In the first case there emerges from the lens a pencil of rays apparently diverging from a point p as in fig. (1) and in the other case a pencil of rays converging to a point p as in fig. (2).

The same will be true of the pencils which emanate from other points in PQ.

The assemblage of points from which, in the one case, the pencils after refraction appear to diverge, or to which in the other case they converge, is called *the image* of the object PQ formed by the lens.

In the former case the image is called a *virtual image*, in the latter a *real image*, these terms being defined as follows.

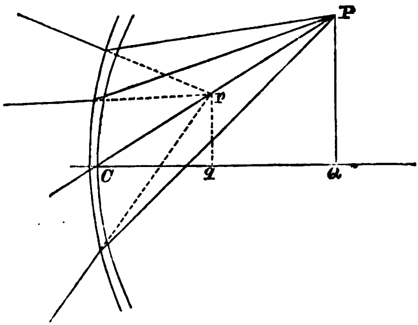
A real image formed by a lens or mirror is an image, through the points of which the pencils of light which form the image do actually pass before diverging from them.

A virtual image is one through the points of which the rays of light do not actually pass.

The image in fig. (1) is called an *erect image*, that in fig. (2) is an *inverted image*.

Secondly if the light from any object fall upon a *concave* lens it is easy to see that a virtual image of the object is formed nearer to the lens than the object. The position

Fig. (3).



of any point p of the image is determined in terms of that of the corresponding point P of the object by the formula

$$\frac{1}{Cp} - \frac{1}{CP} = \frac{1}{f},$$

C being the centre of the lens, and p being on the line CP.

In a similar way the formation of an image by a mirror can be explained (Art. 47).

78. In any of these cases an eye suitably placed so as to receive the pencils of light after divergence from the points of the image will be rendered sensible of the apparent existence of an object in the position of the image.

This image will more or less closely resemble the original object. It has however two defects.

(1) Indistinctness, arising from the fact that the pencils which emanate from various points in the original object do not accurately converge to or diverge from *points* after refraction through the lens; the formulæ we have used being only approximations. The image will thus consist of a number of small overlapping circles or ovals, which will cause the general appearance to be somewhat hazy. With good lenses, if the curvatures of their surfaces be not very large, this defect is not very serious, and can be

somewhat alleviated by a proper choice of the form of the lens.

(2) Curvature. It is clear that the formula

$$\frac{1}{Cp}-\frac{1}{CP}=-\frac{1}{f}$$

will not give Cp in a constant ratio to CP. Hence for instance if PQ be a straight line the image pq will not be a straight but a curved line.

The image of any object will similarly be differently curved from the object itself.

The image is also rendered indistinct and imperfect by the fact that white light is composed of a great number of kinds of light of unequal refrangibility. This subject is treated of in Chapter IX.

79. The preceding Articles furnish a ready means of ascertaining by experiment the focal length of a convex lens. If it be placed so as to form a *real* image q, of any bright object Q in its axis, and the distances of the point and its image from the lens be measured, the focal length is known from the formula which applies to fig. (2) of Art. 77,

$$\frac{1}{f}=\frac{1}{CQ}+\frac{1}{Cq},$$

where f is the numerical value of the focal length.

It can be more readily found by the following method.

Since $$\frac{1}{CQ}+\frac{1}{Cq}=\frac{1}{f},$$

we have $$CQ+Cq=Qq=\frac{CQ\,.\,Cq}{f}.$$

Hence Qq the distance between the point and its image is least when $CQ\,.\,Cq$ is least, or when $\frac{1}{CQ}\cdot\frac{1}{Cq}$ is greatest. But it is known that when the sum of two quantities is con-

stant their product is greatest when they are equal. Hence $\frac{1}{CQ} \cdot \frac{1}{Cq}$ is greatest when

$$\frac{1}{CQ} = \frac{1}{Cq} = \frac{1}{2f}, \text{ or } CQ = Cq = 2f.$$

Hence the least value of Qq is $4f$.

Let then a candle-flame or other light be fixed, and the lens be placed just a little farther from it than the focal length; a real image of the light will be formed at a great distance off, which can be received on a screen. Move the lens slowly away from the light. The image will be found to approach nearer to the light till the lens has been moved to such a position that it is just about as far from the candle as it is from the image received on the screen. If the lens be moved still farther from the light, it will be found that the image begins to recede.

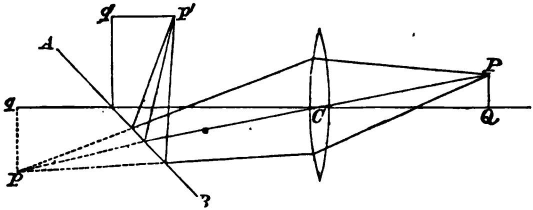
By the above investigation, if we measure the distance between the light and its image in their nearest position, one quarter of that distance will be the focal length required.

80. A good illustration of the formation of a real image is furnished by the Photographic Camera.

This consists essentially of a box with a lens fixed at one end; at the other end of the box is placed a screen. The axis of the lens being directed to the object whose photograph is required, an inverted image of the object is formed within the box. On placing the screen to coincide with this image, a distinct inverted representation of the object is seen on the screen, and if for the screen be substituted a piece of glass properly prepared, the chemical action of the light on the substances with which the glass is covered will leave an accurate delineation of the lights and shadows of the original object.

81. Instead of receiving the inverted image, formed as in the last Article, immediately on a screen, let the rays of light be caught, before they converge to form the image, by a plane mirror placed within the box and inclined to the axis of the lens at an angle of 45°. The image will then

be distinctly formed on a screen placed in a proper position in a plane parallel to the axis of the lens.



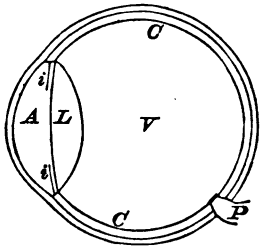
Thus, if QC be the axis of the lens, AB the mirror, the pencil of light emanating from P would after refraction through the lens be made to converge to a point p in PC produced.

The mirror AB intercepts the light before it reaches p, and causes it to converge to p', a point as far in front of AB as p is behind.

Thus an image $p'q'$ of PQ will be formed, and will be distinctly visible on a screen of oiled paper or thin ground glass placed in the top of the box.

Such an instrument is called a Camera Obscura.

82. The Eye itself furnishes another illustration of the formation of a real inverted image.



The figure represents a section of the right eye by a plane through the optic nerve. The external boundary

of this consists of portions of two spherical surfaces, the larger and back part being opaque, the smaller spherical portion which is in front being protuberant and transparent. The back part is called the Sclerotic membrane; the front part, which is set in the sclerotic like a watch-glass in its rim, is called the Cornea.

Within the sclerotic is another membrane called the Choroid membrane. This covers nearly the whole of the internal surface of the sclerotic, having a circular opening in front. To the border of this circular opening is attached a membranous ring called the Iris, having its plane perpendicular to the line joining the centres of the cornea and the sclerotic. The external face of this ring is of various colours, and the aperture in its centre, called the Pupil, is capable of being enlarged or diminished, so as to admit more or less light.

Behind the iris, and with its axis in the line joining the centres of the cornea and sclerotic, is placed a transparent gelatinous substance of the form of a double convex lens, called the Crystalline lens. The face of this lens towards the anterior surface of the eye is flatter, its posterior surface more convex.

The space between the cornea and this lens is filled with a transparent fluid called the Aqueous humour, that between the crystalline lens and the back of the eye by a transparent fluid called the Vitreous humour.

At the back of the eye the Optic Nerve enters through the sclerotic and choroid membranes, and forming a slight protuberance within the latter, spreads out over nearly its whole extent into a delicate tissue of nerves called the Retina.

The refractive indices of the aqueous and vitreous humours are nearly equal to that of water, that of the crystalline lens is somewhat greater.

The whole of the inner surface of the choroid membrane is coloured a deep brown or black and is totally incapable of reflecting light.

If an object of any kind be placed in front of the eye, the rays from any point of it are incident on the cornea,

and are refracted by the aqueous humour, the crystalline lens, and the vitreous humour, so as to converge to a point on the retina.

The eye thus optically resembles a photographic camera. A real inverted image of objects in front is formed on the retina of the eye, and by means of the optic nerve the impression is conveyed to the brain.

This impression remains for a short time after the light which forms it is withdrawn. Thus, for instance, if a bright point be whirled about in a circle, the eye will see it in each position for a short time after it has left that position, and if the point move with sufficient velocity the eye will see a ring of light.

83. The eye is capable of being changed slightly in form by the action of certain nerves, so as to bring rays which come from points at different distances accurately to a focus on the retina.

This adjustment can be effected at will within certain limits which vary for different eyes.

Some eyes can only be conveniently adjusted to view near objects, as they require considerable divergence in the pencils which they can refract accurately to a point on the retina; less divergent pencils being refracted by them to a focus in front of the retina.

In order to see distinctly objects at a greater distance, such eyes make use of concave lenses which form a virtual image of the object, nearer to the eye than the object itself (Art. 77). Such eyes are called short-sighted.

Some eyes, on the other hand, can only refract pencils having slight divergence, such as those which come from points at a great distance, accurately to points on the retina; more divergent pencils being refracted to points behind the retina.

Such eyes are called long-sighted. To enable them to see nearer objects, a convex lens is employed which forms a virtual image of any object, at a greater distance than the

object, the pencils from points of which will not be too divergent to be refracted to a focus on the retina (Art. 77).

These remarks explain the use of spectacles.

84. A real image of any object looked at is formed in each eye of the observer. These images differ slightly from each other, as the positions of the two eyes with respect to any tolerably near object sensibly differ. Thus, in looking at a solid object, such as a round pillar, the right eye sees more of the right side, the left eye sees more of the left side of the pillar. By combining the impressions produced on the mind by these two images we obtain an idea of the solidity of objects viewed, as also of their comparative distances.

The common Stereoscope is an application of the same principle. Two photographs are taken of the same object from points of view a little distance apart. These photographs are so placed that the right eye looks only at that one which was taken from a point of view most to the right, while the left eye looks at the other: by this means an appearance of relief can be imparted to pictures of solid objects, such as statuary, trees, or buildings.

This appearance can be produced even in the pictures of objects which are too distant to appear differently to our two eyes. For instance, if photographs of the moon be taken at two different times, on one of which, owing to libration in longitude, more of its east limb is visible, and on the other more of its west limb, and the two photographs be placed in a stereoscope, the two pictures combined will produce the impression of a solid globe of very small dimensions.

85. No eyes are capable of seeing distinctly objects placed very close to them. Consequently it is impossible for an unaided eye to obtain a clear view of a very small object; for such an object will not subtend a sensible angle at the eye, unless it is placed too near to the eye for the latter to be able to refract the rays emanating from any point of the object accurately to a focus on the retina.

In such a case a convex lens placed close to the eye, between the eye and the small object, will produce a virtual image of the object behind the lens (Art. 77, fig. 1), which will subtend the same angle at the centre of the lens as the object, and yet may be made to appear at any suitable distance.

Thus, by the use of such a lens, a small object may be magnified, that is, an image of it may be formed at such a distance as to be distinctly visible and subtending a sensible angle at the eye.

Such a lens, employed for this purpose, is called a simple Microscope, and is used in botanical and other investigations.

A combination of two lenses having a common axis, and placed a little distance apart, can be used for the same purpose: the first lens forming a virtual magnified image, which is again magnified by the second.

In either case the magnifying power is fairly measured by the ratio of the angle subtended at the eye by the final image to the angle which would be subtended by the object at the eye, if the object were placed at a distance from the eye equal to that of the final image. This ratio is easily seen to be the ratio of the linear magnitude of the final image to that of the object.

In order to magnify or examine minutely a *distant* object we cannot use exactly the same method.

The following is the general plan on which Optical Instruments for this latter purpose are constructed:

A lens or mirror, called the Object-glass or Mirror, is first used to produce a real image of the distant object. This image is close at hand, and can be examined and magnified by a lens as if it were a real object.

Such instruments are called Telescopes. Their construction is explained in the next Chapter.

EXAMPLES. CHAPTER VII.

1. A convex lens, of focal length one-fifth of an inch, is used as a simple microscope by an eye which sees most distinctly at a distance of 14 inches; find the magnifying power.

2. A long-sighted person, who can see most distinctly at a distance of two yards, uses glasses of two feet focal length; at what distance from the glasses should the object be placed? and how much will it be magnified?

3. Shew that when an image of any object is formed by a fixed convex lens, there are two positions of the object for which the size of the image is m times the size of the object, and that the distance between the two positions is $\frac{2f}{m}$, f being the focal length of the lens.

4. A screen, placed at right angles to the axis of a lens, receives the image of a small object. If the magnifying power of the lens in this position be 20, then the distance of the lens from the screen is 21 times the focal length.

5. A bright point is placed on the axis of a convex lens, so that the distance between the point and its image is the least possible: prove that if a concave lens of the same focal length be introduced half-way between the bright point and the convex lens, the image will be moved half as far again from the lens.

6. A convex lens is held so that the distance between a bright point and its image is the least possible: two other lenses are then introduced, one half-way between the lens and the bright point, and the other half-way between the first lens and the image. If the image formed by refraction through the three lenses have the same position as the former image, prove that the sum of the focal lengths of the three lenses is algebraically zero.

7. A double convex lens of focal length f is at a distance b from a plane mirror, and the axis of the lens is perpendicular to the mirror; shew that if a man places his eye at a distance from the lens $\frac{f(2b-f)}{2(f-b)}$ on the other side from the mirror he will see the image of his eye by parallel rays.

8. A person who reads small print at a distance of two feet finds that with a pair of plano-convex spectacles he can read it at a distance of one foot; find the radius of the curved surface, the refractive index of glass being $\frac{3}{2}$.

9. If a convex lens be placed between any luminous object and a screen, find the position of the lens that a real image may be formed on the screen. Prove that there are two such positions, and that if m_1, m_2 be the ratios of the image to the object in these two positions, $m_1 m_2 = 1$.

10. When an object is placed before a convex lens whose focal length is f, at a distance $\frac{3}{2}f$ from the lens, shew that the image is twice the object in linear dimensions. When the object is placed at double this distance from the lens, shew that the length of the image is one-half the length of the object.

11. The image of a very distant object is formed by a convex lens; a plane mirror is placed at a distance from the lens equal to $\frac{3}{2}$ the distance of the image from the lens, perpendicular to the axis; shew that a second real image will be formed by the reflected light on the other side of the lens, of the same linear dimensions as the first.

12. An object is placed at a distance c in front of a convex lens whose focal length is a, c being greater than a. A concave mirror whose focal length is b is placed at a distance a behind the lens. An image of the object is formed by rays which are refracted through the lens, reflected at the mirror, and again refracted through the lens. Shew that this image is at a distance $\frac{a^2}{b} + c - 2a$ behind the lens, and that it is equal in magnitude to the original object, but inverted.

13. A magnifying-glass consists of two convex lenses whose thickness, as also the distance between them, may be neglected. When used by a person who can see most distinctly at a distance of eight inches, the ratios of the magnifying powers of the first lens alone, the second alone, and the two combined are as $3:4:5$. Find the focal lengths of the lenses used.

CHAPTER VIII.

ON COMPOUND OPTICAL INSTRUMENTS.

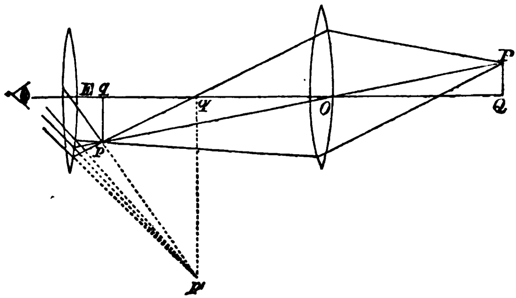
86. TELESCOPES are of two kinds:

Refracting Telescopes, in which a convex lens is employed to produce an image of the distant object.

Reflecting Telescopes, in which this image is produced by means of a concave mirror.

87. The most important kind of Refracting Telescope is usually called the Astronomical Telescope.

In its simplest form it consists of two lenses, placed so as to have a common axis, the lens which is nearest to the object viewed, and which is called the Object-glass, having a much greater focal length than the other, which is called the Eye-glass.



Let O be the centre of the object-glass, E that of the eye-lens, PQ the object to be looked at, which is usually at a very great distance from the object-glass. The rays from any point P of this object will be incident centrically on the object-glass, and will converge to a point p in PO produced (as in Art. 77, fig. 2). Thus a real inverted image pq of the object is formed.

The eye-lens receives the pencil of rays which diverges from the point p and refracts them so as to make them diverge from some point p' on Ep produced. (Art. 74.)

A virtual image $p'q'$ of the object will thus be formed, which can be seen by an eye placed close to the eye-lens.

By altering the position of the eye-lens the position of $p'q'$ may be altered until it is at a convenient distance for the eye to see it. It is usual to assume $p'q'$ at an infinite distance, so that the rays emerge parallel from the eye-lens. In viewing objects at a very great distance, such as the moon or stars, the eye will probably adjust itself so as to be fitted for receiving parallel rays, and if so, the above assumption will be correct.

In this case Eq will be the focal length of the eye-glass, and if PQ be very distant from O, Oq will be the focal length of the object-glass.

Thus OE will be the sum of the focal lengths of the lenses.

The eye-glass and object-glass are fixed in two tubes, of which the eye-glass tube slides in the other to permit of proper adjustment for different eyes.

88. The magnifying power of such a telescope will be fairly measured by the ratio of the angle subtended at the eye by the final image of any object to the angle subtended at the eye by the object itself.

If the distance of the object be very large compared with the length of the telescope, the angle subtended by the object at the eye of the observer will not appreciably differ from that subtended by it at the centre of the object-glass.

Thus, supposing the eye placed close to the eye-lens, the magnifying power will be appreciably

$$=\frac{\angle pEq}{\angle POQ}=\frac{\angle pEq}{\angle pOq}$$
$$=\frac{\tan pEq}{\tan pOq},$$

since the angles are all small in practice,

$$=\frac{\dfrac{pq}{Eq}}{\dfrac{pq}{Oq}}=\frac{Oq}{Eq}=\frac{\text{focal length of object-glass}}{\text{focal length of eye-glass}},$$

taking the approximate values of the last Article for Oq and Eq.

89. The office of the object-glass is thus to form near at hand an image of the distant object. The eye-glass is used to magnify this image just as if it were a real object. (Art. 85.)

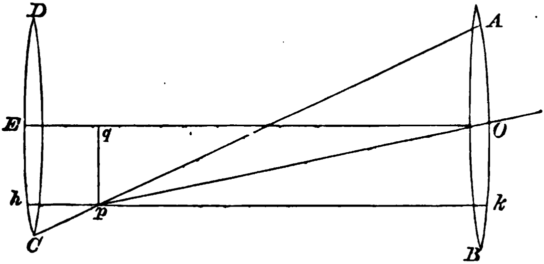
This image differs however from a real object in that the rays of light do not diverge in all directions from the various points of it. The rays from p, for instance, only proceed in the lines in which rays have come to p from some point of the object-glass.

Thus, if the eye were simply placed to view pq without the intervention of an eye-glass, not only would it have to be placed farther from pq in order to ensure distinct vision, and pq would thus appear to subtend a smaller angle at the eye, but the pencils from points in pq a short distance away from the axis OE would not come into the eye at all.

The image would thus not only not be magnified, but a much smaller portion of it would be seen than is the case when an eye-glass is used. Thus a second advantage of the eye-glass is, that the pencils from points outside the axis are bent round so as to enter an eye placed close to the eye-glass.

The amount of the object that is visible through the eye-lens is termed *the field of view.* It will evidently be circular in form, and the angle subtended by its radius at the eye can be easily ascertained.

It is clear that any point p of the image will be distinctly seen, if the whole of the pencil which converges to form it is incident on the eye-glass.



This will be the case if the lowest ray, after passing through p, is incident on the eye-lens; thus the point farthest out of the axis, which is visible by a whole pencil, is a point p, such that the ray of its pencil which comes from the top of the object-glass just comes to the bottom of the eye-glass.

Let now $EC = y_e$ = half aperture of eye-glass,

$AO = y_o$ = half aperture of object-glass,

$Eq = f_e$ = focal length of eye-glass,

$Oq = f_o$ = focal length of object-glass.

Drawing a line hpk through p parallel to EO, we get by similar triangles

$$Ch : hp :: Ak : kp ;$$

$$\therefore\ y_e - pq : f_e :: y_o + pq : f_o ;$$

$$\therefore\ f_e y_o + f_e \,.\, pq = f_o y_e - f_o \,.\, pq ;$$

$$\therefore\ pq = \frac{f_o y_e - f_e y_o}{f_o + f_e},$$

and the angular radius of the field of view

$$= \frac{pq}{qE} = \frac{f_o y_e - f_e y_o}{f_e(f_o + f_e)}.$$

90. It is easy to see that from a point a little below p, a portion only of the pencil will fall upon the eye-lens, this portion getting less and less until, from points beyond a certain distance from the axis, no part of the pencil will reach the eye-lens.

There will thus be a ring of points imperfectly seen, surrounding the distinct field of view. This ring is known as the Ragged Edge. It is usually destroyed by a material ring placed so as to stop all the rays of the pencils of which part only would fall on the eye-lens.

The radius of the aperture of this ring is clearly the value of pq given in the last Article.

It will be noticed that the pencils from different points of the image pq do not fill up the whole of the eye-glass. They are thus incident excentrically on the eye-glass in conformity with the remark of Art. 72.

91. The axes of the small pencils which fall on the eye-piece, and by means of which the image is finally seen, are bent in passing through the eye-lens.

It is found, when the calculations of the form of a pencil, after excentrical refraction through a lens, are carried to a higher approximation, that our assumption, that the virtual image p' of any point p lies on the line Ep, is not absolutely correct, and that it becomes less and less correct the farther the point p is distant from the axis.

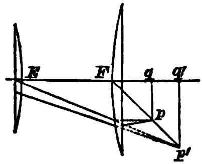
The image $p'q'$ actually seen is thus distorted from the shape of pq, and *ceteris paribus* it is found that this distortion increases as the focal length of the eye-lens decreases.

To remedy in some measure this evil, a combination of two lenses placed at a short distance from each other is often employed instead of a single lens.

The axis of the excentrical pencil is thus bent round at one lens and again bent round at the other, and it is found by experience and by calculation that the distortion produced in this way is much less than would be produced by a single lens equivalent to the combination of the two lenses.

92. Two such combinations have been specially used. In the first, known as Ramsden's Eye-piece, the two lenses are of equal focal length, and the distance between them is equal to two-thirds of the focal length of either.

The lens nearest to the eye is called the Eye-lens, and the other the Field-glass.



Thus let pq be the image produced by the object-glass and let f be the numerical value of the focal length of either lens. Let F and E be the centres of the two lenses. The field-glass will form a virtual image of pq in a position $p'q'$, such that

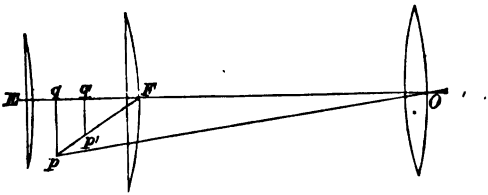
$$\frac{1}{Fq} - \frac{1}{Fq'} = \frac{1}{f}.$$

This image, in accordance with the assumptions explained in Art. 87, will be nearly at the principal focus of the eye-glass. Thus $Eq' = f$, and since $EF = \frac{2}{3}f$, we must have $Fq' = \frac{1}{3}f$, whence $Fq = \frac{1}{4}f$.

The eye-piece will therefore be placed so that the image formed by the object-glass is nearly at a distance $\frac{1}{4}f$ in front of the field-glass.

93. In the other combination, known as Huyghens' Eye-piece, the focal lengths of the lenses are in the ratio of 1 to 3, the distance between them being equal to the difference of their focal lengths.

Let E and F be the centres of the eye-lens and field-glass, f and $3f$ the numerical values of their focal lengths. Then $EF = 2f$.



The eye-piece is so placed that the rays which, after refraction through the object-glass, converge to form any point p of the image are incident on the field-lens before reaching p.

They are thus made to converge approximately to some point p' in the line Fp, and a real image $p'q'$ is formed by the field-lens.

This image must be nearly in the principal focus of the eye-lens, and must therefore be half-way between E and F.

We have also

$$\frac{1}{Fq'} - \frac{1}{Fq} = \frac{1}{3f};$$

whence, since $Fq' = Eq' = f$, we get $Fq = \frac{3f}{2} = \frac{3}{4} FE.$

A second important advantage of Huyghens' eye-piece will be pointed out hereafter (Art. 117, end).

94. The image of the distant object formed by the object-glass being formed by centrical pencils, the angle subtended at the centre of the object-glass by any part of the object is equal to that subtended at the same point by the corresponding part of the image.

The latter angle, and consequently the former, can be deduced if we measure the linear magnitude of the part of the image, the distance of this image from the object-glass being known.

The image formed by the object-glass being, if we use a single lens or Ramsden's eye-piece, a real image; if a piece

of glass with lines ruled on it at equal distances be placed to coincide with the image, it is clear that the lines on the glass will be distinctly seen through the eye-piece along with the image.

Thus the distances of points on the image from each other could be ascertained by noticing the number of intervals on the glass scale between them.

A better arrangement is to place in the field of view a framework carrying one or more fine wires or spider threads, which can be moved along the framework by means of a screw.

If the framework be so placed that the plane of the moveable wires coincides with the real image, the distance between any two points of the image can be measured by moving the wire from one point of the image to the other, and noting how many turns of the screw are required to effect this displacement. The distance between two consecutive threads of the screw being known, the required distance is thus obtained.

The obliquity of the pencils which are refracted by the object-glass is in practice always very small. The angle subtended at the centre of the object-glass by the line joining two points of the image is thus nearly proportional to the linear distance of the points from each other, and its circular measure will be nearly equal to that distance divided by the focal length of the object-glass.

Huyghens' eye-piece cannot be used for such measurements, as the image given by the object-glass is never actually formed.

95. The astronomical telescope gives us thus the means of viewing any distant object under a much greater angle than we could view it with the naked eye. It has also another advantage.

In viewing any object with the naked eye, only so much light comes from each point of the object as will fill the pupil of the eye.

In viewing the same object with the telescope, such an amount of light comes into the eye from each point of the

object as when originally proceeding from that point fills the object-glass.

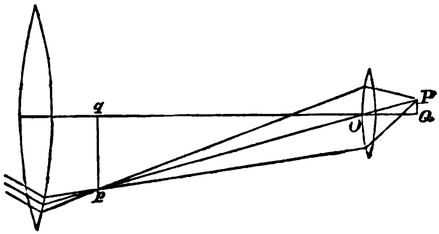
Thus much more light will enter the eye in the latter case than in the former. A telescope enables us, for instance, to see stars which are so faint that the naked eye cannot distinguish them; and the larger the object-glass in diameter the more will this effect be produced.

96. The only condition necessary that a convex lens shall form *a real* image of an object in front of it is, that the object shall be farther from the lens than its principal focus (Art. 77).

The astronomical telescope may thus be modified so as to view near objects, if the focal length of the object-glass be diminished.

It then becomes a *Compound Microscope.*

Let O be the centre of the object-glass and let PQ be the object, OQ being a little greater than the focal length of the object-glass.



An inverted real image pq is formed by the object-glass, which image is again viewed and magnified by the eye-lens.

In practice the conditions required to be satisfied by a microscope are so different from those of a telescope that such an instrument as the above would be of little real value.

The pencils from any point of the object are so divergent the object being near to the object-glass, and for points of

the object outside the axis so obliquely incident, that the approximations of the previous chapters of this treatise fail accurately to represent the facts.

In the compound microscopes which are actually used, a series of three lenses placed near together is substituted for the single lens O, and instead of a single eye-lens a compound eye-piece of two lenses is used.

The theoretical investigation of the proper forms of the lenses involves considerations into which this book cannot enter.

97. The compound microscope of the last Article produces an inverted image of the object placed in front of it. If then, instead of a single lens, or Ramsden's or Huyghens' eye-piece, a compound microscope be applied to view the real image formed by the object-glass of the telescope in Art. 87, the image formed by the object-glass which is inverted, will be inverted again, and the eye will see an erect image of the original object.

Such an arrangement is called an erecting eye-piece, and is applied to telescopes used for observation of terrestrial objects. The ordinary erecting eye-piece consists of four lenses, two of which may be considered as the object-glass of the compound microscope, and the other two as the eye-piece.

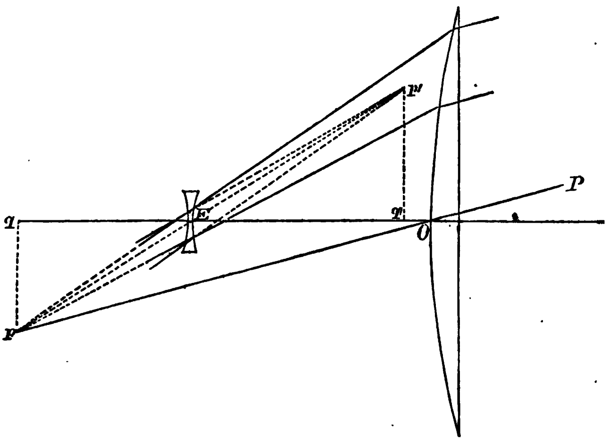
98. The second kind of refracting telescope is usually known as Galileo's Telescope, from the name of the inventor.

A double convex lens is employed as before to form an image of the distant object.

A concave lens is placed with its axis in the same line as that of the object-glass so as to catch the rays before they have converged to form the image.

Thus a pencil from a point P of the object, falling on the object-glass, would be refracted to a point p in PO produced, O being the centre of the object-glass.

Some portion, not the whole, of this pencil is caught by the concave lens E, which is so placed that the pencil con-



verging to p is made to diverge from a point p' in pE produced. This will be the case if Eq be not less than the focal length of the lens E.

The pencil will then enter an eye placed close to E, as if it came from p', and the eye will see an erect image of PQ at $p'q'$.

The lens E will be adjusted so that this image is at a convenient distance for distinct vision.

If we make the assumption that $p'q'$ is at an infinite distance, and PQ also at an infinite distance, as in Art. 87, it is easy to see that Oq is the focal length of the object-glass, and Eq that of the eye-glass, so that OE is the difference of the focal lengths of the object and eye-glasses.

99. The field of view in this telescope is limited by the object-glass, and not by the eye-glass as in the Astronomical Telescope. For it is clear that the pencil from every point that is clearly seen fills up the whole of the eye-glass, while it is only a portion of the pencil that actually falls on the object-glass which is used in producing vision.

The refraction of the pencils is thus centrical at the eye-glass and excentrical at the object-glass as far as the rays actually useful are concerned.

The field of view can be calculated as in the case of the Astronomical Telescope.

The lowest point of the image pq that is seen by a full pencil is the point which is formed by a pencil whose highest ray passes through the top of the object-glass, and consequently the distance of this point from the axis and the angular magnitude of the field of view can be obtained by similar triangles, just as in Art. 89.

If y_0, y_e are the radii of the apertures of the object and eye-glass respectively, and f_0, f_e the focal lengths of these lenses; with the assumptions at the end of the last article, we have, by similar triangles,

$$\frac{y_e + pq}{f_e} = \frac{y_0 + pq}{f_0};$$

$$\therefore \quad pq = \frac{f_e y_0 - f_0 y_e}{f_0 - f_e},$$

and the angular radius of the field of view

$$= \frac{pq}{f_e} = \frac{f_e y_0 - f_0 y_e}{f_e(f_0 - f_e)}.$$

In order to ensure a large field of view the aperture of the object-glass must therefore be considerable.

The magnifying power of Galileo's Telescope will also easily be found to be expressed by the ratio of the focal length of the object-glass to the focal length of the eye-glass.

The chief advantage of Galileo's Telescope is, that it gives an erect image with the use of only two lenses.

It cannot be used for measurements, as the image formed by the object-glass is a virtual one.

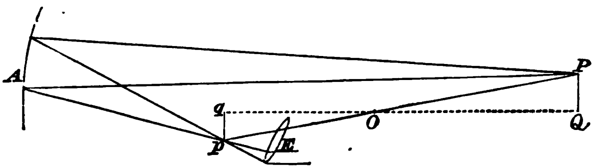
The ordinary Opera-glass consists of a pair of Galilean Telescopes placed with their axes parallel.

100. We have now to consider the second class of Telescopes, in which the image of the distant object is formed by a reflecting spherical surface.

The different kinds of reflecting telescopes chiefly differ in the arrangements made for viewing and magnifying this image.

We shall first describe, as the simplest form, and the one which has been used for the largest reflecting telescopes, Herschel's construction.

101. Herschel's Telescope consists essentially of a large concave mirror.



Let A be the centre of the face of the mirror, O the centre of the spherical surface of which it is formed.

Let PQ be any distant object.

The rays from any point P of this object will fall upon the mirror, and be approximately reflected to some point p, in PO produced. This assumption is equivalent to taking the secondary focus of the reflected pencil as the point of convergence of the whole pencil, an assumption which will not be far wrong if, as is always the case in practice, the obliquity of the pencil from P be very small.

Thus a real inverted image of PQ will be formed at pq.

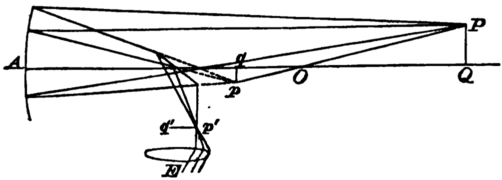
The rays which have converged to form the points of this image are then received on a convex lens of short focal length, by which a larger virtual image of pq is formed, as in the Astronomical Telescope.

This lens is placed towards one side of the tube, at one end of which the large mirror A is fixed, and with its axis

slightly inclined to that of the large mirror. The light from the point P is thus not intercepted to any great extent by the observer's head, but the eye is only able to perceive parts of the image which have been formed by slightly oblique pencils.

The reader who has carefully studied and understood the investigation of the field of view and magnifying power in the case of the Astronomical Telescope will have no difficulty in investigating similar formulæ for Herschel's Telescope.

102. A slightly different construction more suitable for small telescopes is known as Newton's Telescope. In



this telescope the rays from any point P are reflected by the large mirror so as to converge to a point p.

An inverted image of the object PQ would thus be formed at pq just as in the last Article.

A plane mirror is placed with its plane inclined at an angle of 45° to the axis of the object mirror, so as to intercept the pencils before they converge to points of pq.

By this mirror the pencil converging to p will be made to converge to p', a point at the same distance in front of the plane mirror as p is behind it.

A real image $p'q'$ of PQ is thus formed, which can be viewed by an eye-lens or eye-piece placed in the side of the tube which carries the large mirror, the axis of the eye-piece being at right angles to that of the large mirror.

The adjustment for distinct vision can be made by attaching the small plane mirror to the eye-piece and

giving them both a motion parallel to the axis of the object mirror.

By moving them towards A the plane mirror is removed farther from pq, and $p'q'$ is thus brought nearer to the eye-lens. By moving them in the opposite direction, $p'q'$ is removed farther from the eye-lens.

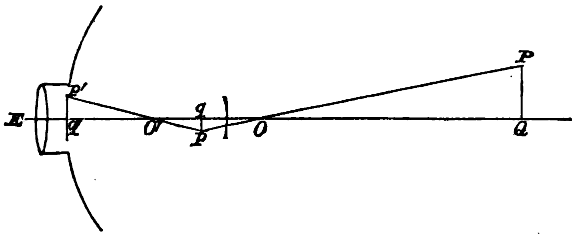
The magnifying power and field of view can be investigated by a similar process to that given for the Astronomical Telescope.

103. Two other forms of reflecting telescope remain to be described, Gregory's and Cassegrain's.

In each of these the large mirror is pierced with a circular aperture in its centre, to receive the eye-piece.

The pencil of rays from any point P of a distant object falls upon the object-mirror and is reflected to converge to a point p; thus an image pq of any object PQ is formed.

Fig. (1).

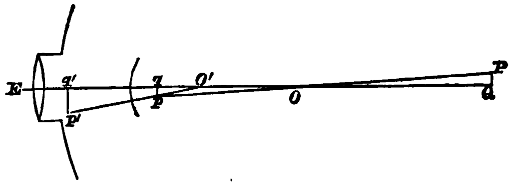


In Gregory's Telescope the pencil after converging to p is received on a small concave mirror whose axis coincides with that of the large mirror, and is reflected so as to converge to a point p' in pO' produced, O' being the centre of the spherical surface of this small mirror. Thus a real erect image $p'q'$ of PQ is formed, and the position of the small mirror is so chosen that this image can be distinctly viewed by the eye-piece E, which is capable of a slight adjustment for this purpose.

In Cassegrain's Telescope the pencil of rays from P is caught before it converges to p by a small convex mirror,

by which it is made to converge to a point p' in $O'p$ produced, O' being the centre of the spherical surface of this small mirror.

Fig. (2).



Thus an inverted real image of the object PQ is formed at $p'q'$, and the small mirror is so placed that this image can be conveniently viewed by the eye-piece.

The investigation of the field of view in these telescopes is too complicated to find a place in this treatise. We will give an approximate investigation of the magnifying power in Gregory's Telescope.

Let F be the focal length of the large mirror, f_m that of the small mirror, f_e that of the eye-piece.

Then approximately with the usual conventions, in Fig. (1)

$$Oq = F, \qquad Eq' = f_e.$$

Also the distance of the image $p'q'$ from the small mirror is in practice very large, compared with the focal length of the small mirror. Hence approximately we may take $O'q = f_m$; whence we get also $O'q' = qq' - O'q = F - f_m$ nearly.

Now the magnifying power is evidently measured by the fraction

$$\frac{\angle p'Eq'}{\angle POQ} = \frac{\angle p'Eq'}{\angle pOq} = \frac{\dfrac{p'q'}{Eq'}}{\dfrac{pq}{Oq}} \text{ approximately,}$$

$$\therefore \quad \text{magnifying power} = \frac{p'q'}{pq} \cdot \frac{Oq}{Eq'} = \frac{O'q'}{O'q} \cdot \frac{Oq}{Eq'}$$

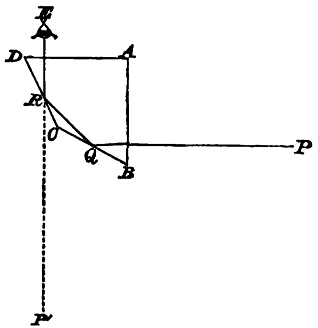
$$= \frac{(F - f_m) \cdot F}{f_m \cdot f_e}$$

$$= \frac{F^2}{f_m f_e} \text{ nearly,}$$

since f_m is small compared with F.

Gregory's construction is frequently used for small reflecting telescopes, and the fact of its giving an erect image is an advantage in viewing terrestrial objects. For large telescopes, the difficulty of supporting the small mirror accurately in its right position without being liable to tremors diminishes its value. For such telescopes the form adopted by Herschel is the best.

104. It is sometimes convenient to be able to turn through a right angle the direction in which the eye sees objects through a telescope or microscope. This can be effected by placing in front of the eye-piece of the telescope or microscope an instrument called the Camera Lucida.



This consists of a glass prism whose section perpendicular to its axis is a quadrilateral figure $ABCD$, one angle of which A is a right angle, the angle C opposite A being an obtuse angle of such a magnitude that a ray of light which enters the prism at right angles to AB shall

after internal reflection at BC and CD emerge in a direction at right angles to AD.

The deviation of this ray being thus a right angle, the acute angle between CB and DC must be half a right angle or BCD must be 135°.

The angles at B and D are equal and therefore each equal to $67\frac{1}{2}$ degrees.

With any ordinary kind of glass it will be found that rays incident on BC perpendicular to AB are incident at an angle greater than the critical angle and are totally internally reflected. The same will happen at CD and thus no light will be lost.

This will also be true for rays slightly inclined to this direction. If therefore a pencil of light emanating from a point P be incident on the lower part of AB, with its axis only inclined at a small angle to the normal to AB, this pencil will emerge from AD with its axis at right angles to its original direction and the eye will see an image of the point P at some point P' in this new direction.

Thus, for instance, if the tube of a microscope be placed horizontally, and the camera lucida be placed close in front of its eye-piece, the pencils of light diverging from different points of the virtual image formed by the eye-piece will, after passing through the camera lucida, give to the eye the impression of a horizontal image of the object viewed through the microscope.

The eye-piece can be adjusted so that this horizontal image shall appear at any required distance, and if a piece of paper be placed below the eye at this same distance and the eye be placed with its pupil only half over the edge of the camera, the paper and the image will be distinctly visible together and will appear to coincide.

A drawing of the image can thus be accurately made.

It can be shewn by calculating the position of the foci after each refraction and reflection that, if the size of the camera be small compared with the distances of P or P' from it, the point P' is at the same distance from the eye as P.

EXAMPLES. CHAPTER VIII.

1. The focal lengths of the object-glass and eye-glass of an astronomical telescope are 15 inches and 5 inches respectively, and their radii 3 inches and 2 inches respectively. Find the radius of the stop which will cut off the ragged edge.

2. The diameters of the eye-glass and object-glass of an astronomical telescope are 1 inch and 6 inches respectively, and their focal lengths 1 inch and 20 inches respectively. The axis is pointed to a rod of infinite length at a distance of 150 feet. Find how much of the rod can be seen in the telescope.

3. Shew that the magnifying power of an astronomical telescope furnished with a Ramsden's eye-piece is $\frac{4F}{3f}$; if F be the focal length of the object-glass, and f that of either lens of the eye-piece.

4. If the object-glass of an astronomical telescope be considered as a luminous object, the eye-piece will form a real image of it. Shew that the magnifying power of the telescope is equal to the ratio of the diameter of the object-glass to the diameter of this real image.

5. If the focus of the eye-glass in a Gregory's telescope be at the centre of the aperture of the large mirror, and d be the distance from the large mirror of the image of the small mirror formed by it; shew that the magnifying power of the telescope may be estimated as $\frac{d}{f}$ where f is the focal length of the eye-glass.

6. A person uses the same lens for the field-glass of a Ramsden's and a Huyghens' eye-piece; prove that the magnifying power of his astronomical telescope when fitted up with the latter is half as great again as when fitted up with the former.

7. The axis of an astronomical telescope is directed to the sun so that a real image of the sun is formed by refraction through the object-glass and eye-glass on a screen held perpendicularly to the axis of the telescope. If a be the diameter of this image, α the apparent angular diameter of the sun, d the

distance of the screen from the eye-piece, and m the magnifying power, shew that $m = \dfrac{a \cot \frac{\alpha}{2}}{2d}$.

8. A Galileo's telescope is adjusted so that a pencil from an object 289 feet distant emerges as a pencil of parallel rays; the focal length of the object-glass is one foot, and of the eye-glass one inch: shew that if the axis is directed towards the sun, and a piece of paper be held 23 inches from the eye-glass, an image of the sun will be formed on the piece of paper. The sun's apparent angular diameter being $\cot^{-1} 120$, what is the size of this image, and is it erect or inverted?

9. The focal length of the object-glass of an astronomical telescope is 20 feet and its aperture 15 inches. The eye-glass has a focal length of one inch and an aperture of half-an-inch. What proportion of the moon's disc can be seen at once in the telescope, the angular apparent diameter of the moon being half a degree?

10. Calculate an expression for the field of view of an astronomical telescope fitted with Ramsden's eye-piece, the apertures of the object-glass and field-lens being given. Find what must be the least size of the eye-lens in order that no light may be lost.

11. An eye can see most distinctly at a distance of a feet. The focal lengths of the object and eye-glasses of an astronomical telescope being f_0, f_e feet respectively, and their semi-apertures y_0 and y_e inches respectively, calculate an expression for the field of view and magnifying power when the telescope is adjusted for distinct vision.

12. The focal length of the object-glass of an astronomical telescope is 40 inches, and the focal lengths of four lenses, forming an erecting eye-piece, are respectively $\frac{3}{2}$, $\frac{1}{8}$, $\frac{3}{8}$ and $\frac{3}{2}$ inches, beginning with the field-lens. The intervals between the first and second, and between the second and third, being one inch and half-an-inch respectively; find the position of the eye-lens and the magnifying power, when the instrument is in adjustment for eyes which can see with parallel rays.

13. An astronomical telescope is fitted with a Ramsden's eye-piece, and is adjusted for distinct vision of distant objects.

A convex lens, whose focal length is f_1, is placed in contact with the object-glass, whose focal length is F. Show that the instrument will remain in adjustment if a concave lens be placed in contact with the field-glass, the focal length of the concave lens being $f\left\{1+\frac{f}{F^2}(F+f_1)\right\}$, where $4f$ is the focal length of the field-glass.

Find the magnifying power in this latter case.

14. The lenses of a common astronomical telescope, whose magnifying power is 16, and length from object-glass to eye-glass $8\frac{1}{2}$ inches, are arranged as a microscope to view an object placed $\frac{5}{8}$ of an inch from the object-glass; find the magnifying power, the least distance of distinct vision being taken to be 8 inches.

15. A Galileo's and an astronomical telescope have object-glasses of equal focal length and aperture. Their eye-glasses have equal focal lengths and they have the same field of view for complete pencils; prove that the diameter of the stop in the astronomical telescope should be half the difference of the breadths of the eye-glasses.

16. An astronomical telescope is adjusted to view an object at an infinite distance and is fitted with a Huyghens' eye-piece; shew that its length is $F+\frac{1}{2}f$ where F, f are the focal lengths of the object-glass and eye-glass.

17. Prove that when a ray of light is incident on a Huyghens' eye-piece parallel to the axis, it suffers an equal deviation at each lens.

Shew that this will be the case with any eye-piece composed of two convex lenses, provided that the distance between the lenses is equal to the difference of their focal lengths.

CHAPTER IX.

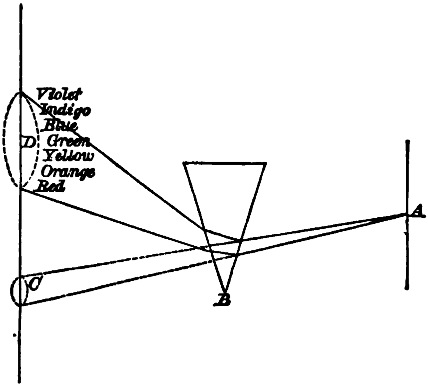
ON DISPERSION AND ACHROMATIC COMBINATIONS.

105. IT has been hitherto assumed that, when a ray of light is refracted out of one medium into another, there is only one refracted ray corresponding to each incident ray.

It was however discovered by Newton that this is not the case with the light of the sun, but that when a ray of sunlight is refracted from air into glass it is separated into a large number of different refracted rays. Newton in effect proved that sunlight is really composed of an infinite number of rays of light of different colours, varying gradually from red through orange, yellow, green, blue, indigo to violet, and of correspondingly different refrangibilities, the index of refraction being least for the red light and increasing by imperceptible degrees, till it becomes greatest for the violet light.

106. Newton's original experiment was conducted somewhat in the following manner:

A is a small hole in the shutter of a darkened room, through which the sunlight comes into the room. From each point of the sun's disc a small pencil will come whose base is the opening A. If A be very small, so that this pencil may be considered to be a single ray, the assemblage of these pencils outside the room will be approximately a conical pencil whose vertex is A, and whose base is the disc of the sun. This cone produced will form a pencil of light with the same solid angle, within the room; and, if it falls



on a screen placed perpendicularly in its path, will form a round patch of white light which is in fact a rough image of the sun.

A prism of glass is interposed so as to receive this pencil near to its edge; and it is then found that if the light be received on a screen, there is formed, not a round patch of white light, but an elongated strip of coloured light, the longer diameter of which is perpendicular to the edge of the prism, and the colours of which proceed from red to violet, the red being the least deviated, and the violet the most.

If the prism be turned about its edge a position can easily be found in which the deviation of the light in passing through the prism is a minimum. This will be the case when the coloured patch, which is called *a spectrum*, assumes the position nearest to that occupied by the white patch when the prism was away.

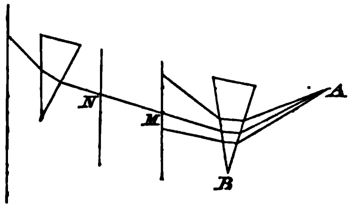
Now, we know, that if the refractive index of a ray be given, the minimum deviation through a prism of a known angle is given. Hence it is not unreasonable to infer that the white light consists of a number of rays of different colours and correspondingly different refrangibilities.

107. Some other experiments were however considered necessary by Newton before he accepted this view. For

instance, placing a second prism with its edge at right angles to that of the first, so as to catch the light after refraction through the first, he found that the spectrum formed on the screen was no broader than before but was shifted sideways, the amount of displacement varying for the different colours being greatest for the violet and least for the red rays.

The following is perhaps the most simple and satisfactory experiment.

The light, after passing through the prism, is received on a screen, with a narrow slit *M* in it parallel to the edge of the prism; all the light therefore, except the small por-



tion which is incident on the slit, is stopped by the screen. At a short distance behind this, another screen, with a slit *N* in it, is placed; and behind this is placed a second prism with its edge parallel to that of the first.

We are thus sure that no light can be incident to the second prism, except in the particular direction *MN*.

By turning the first prism round its edge, we can make all parts of the original spectrum pass in succession over *M*, and this causes the red, orange, &c. light in succession to fall on to the second prism, all at the same angle of incidence.

On performing this experiment, it will be found that when the red light and violet light are thus in succession incident at the same angle on the second prism, the red light is not so much bent as the orange, the orange not so much as the green, and so on, the violet being most refracted.

From these and similar experiments it is concluded that white sunlight consists of an infinite number of kinds of

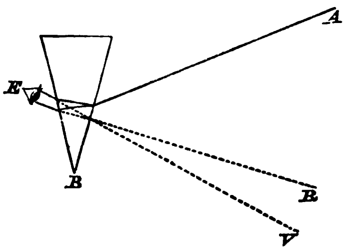
light of different colours and refrangibilities, the red being least refracted and the violet most refracted.

It is found that the same is true of the light emitted by a candle, a burning coal, or any glowing heated body in a solid or fluid state.

108. It remains to show how to separate completely the different kinds of light of which white light is composed. This, it will be observed, is not effected in Newton's fundamental experiment, because each kind of light produces a circular spot of light on the screen, and the circles corresponding to different colours will overlap each other.

Let A, as before, be a small hole or, better still, a narrow slit parallel to the edge of the prism through which light comes from the sun or other luminous body.

When the small pencil from A falls on the prism near its edge, if the prism be placed in such a position that the



deviation is a minimum, we know that the rays of the pencil, after refraction through the prism, approximately diverge from a point at the same distance from B as the original point of light A (Art. 65, end).

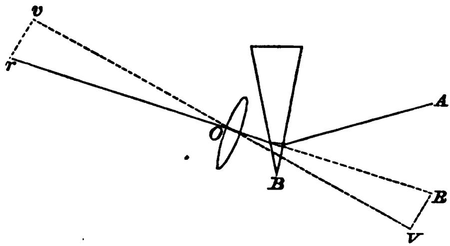
This will be true for light of each kind of refrangibility, but the points will be different for each kind of light. Thus a series of virtual images of A will be formed all nearly at the same distance from B as A is, R the red image being highest, and V the violet image being lowest.

If an eye be placed close to the edge of the prism so as to receive the light after passing through the prism, it will see this series of virtual images, and if the hole or slit A be very small, the eye will thus see a *pure spectrum*, that is a spectrum in which the colours are unmixed.

109. Instead of placing the eye close to the prism, it is better to place an astronomical telescope so as to receive the rays after refraction through the prism.

The object-glass of this telescope will receive the pencils proceeding from various points of the virtual spectrum VR, and will form a real inverted image of VR, which can be viewed by the eye-piece, and measurements of the lengths of its different parts made by the method explained in Art. 94.

Instead of using the eye-piece, a screen may be placed in the position of the real image formed by the object-glass. A pure spectrum will thus be formed on this screen which can be inspected at pleasure.



If the pencil be refracted through a number of prisms, the dispersion, as it is called, of the pencils of different colours is increased by each successive prism, and a much longer pure spectrum is formed than can be obtained with a single prism.

110. An apparatus specially adapted for forming and viewing the pure spectrum of the light from any source is called a spectroscope.

It consists essentially of three parts. First, a tube closed at one end with the exception of a fine slit, through which the light is admitted; secondly, of a prism or series of prisms so placed that the light which has come through the slit and down the tube shall be refracted through them all at an angle of minimum deviation; and, thirdly, of a telescope placed so as to receive the light after refraction through these prisms.

The whole is usually mounted on a stand and is provided with the means of measuring the deviation of any particular ray.

In order to obtain great length of spectrum it is essential to use a large number of prisms. It is clear however that the diverging pencil of light of any particular colour, which comes originally from the slit, would in this case have attained a considerable breadth before its incidence on the last prism, and probably a considerable portion of the light would be finally lost as well as indistinctness produced in the final image by reason of the very different lengths of glass traversed by different portions of the same pencil. To obviate this, a lens is placed in the tube which carries the slit, so that the slit is in its principal focus. The diverging pencil emanating from each point of the slit is thus reduced to a pencil of parallel rays, the width of which for each colour will not increase, as it passes through successive prisms.

This construction is equivalent to removing the virtual images $V...R$ of the slit, in the last article, to an infinite distance.

Spectroscopes are also constructed, in which either by total internal reflection, or by deviation in opposite directions through prisms with different dispersive powers (Arts. 114, 115), there is considerable dispersion without any deviation. These are known as direct vision spectroscopes, and are very convenient for many observations.

111. The angle between the axes of the pencils which converge to v and r respectively can be easily deduced in terms of the angle of incidence of the axis of the incident pencil and the refractive indices for red and violet rays.

Let ϕ be the angle of incidence, μ the index of refraction for mean rays. Let ϕ', ψ', ψ have their usual meaning for mean rays, and let ϕ'_v, ψ'_v, ψ_v denote the same quantities for the violet rays, ϕ'_r, ψ'_r, ψ_r, for red rays. Let μ_v, μ_r be the refractive indices, and D_v, D_r the deviations for these rays respectively.

Then by Art. 60,

$$D_v = \phi + \psi_v - i,$$

$$D_r = \phi + \psi_r - i,$$

$$\therefore\ D_v - D_r = \psi_v - \psi_r$$

But $\sin\psi_v = \mu_v \sin\psi'_v = \mu_v \sin(i - \phi'_v)$

$$= \mu_v \sin i \,.\, \cos\phi'_v - \mu_v \sin\phi'_v \cos i,$$

$$= \mu_v \cos\phi'_v \,.\, \sin i - \quad \sin\phi \,.\, \cos i.$$

Similarly $\sin\psi_r \quad = \mu_r \cos\phi'_r \,.\, \sin i - \quad \sin\phi \,.\, \cos i;$

$\therefore\ \sin\psi_v - \sin\psi_r \quad = \delta(\mu_v \cos\phi'_v - \mu_r \cos\phi'_r)\sin i.$

But $\mu_v \sin\phi'_v = \sin\phi;$

$$\therefore\ \mu_v \cos\phi'_v = \sqrt{\mu_v^2 - \sin^2\phi}.$$

Similarly $\mu_r \cos\phi'_r = \sqrt{\mu_r^2 - \sin^2\phi};$

$$\therefore\ 2\sin\frac{\psi_v - \psi_r}{2} \,.\, \cos\frac{\psi_v + \psi_r}{2}$$

$$= \sin i\,\{\sqrt{\mu_v^2 - \sin^2\phi} - \sqrt{\mu_r^2 - \sin^2\phi}\},$$

$$= \frac{\sin i\,(\mu_v^2 - \mu_r^2)}{\sqrt{\mu_v^2 - \sin^2\phi} + \sqrt{\mu_r^2 - \sin^2\phi}}.$$

Now $\mu_v^2 - \mu_r^2 = (\mu_v - \mu_r)(\mu_v + \mu_r) = 2\mu(\mu_v - \mu_r).$

Also $\psi_v + \psi_r = 2\psi$ nearly,

and $\sqrt{\mu_v^2 - \sin^2\phi} + \sqrt{\mu_r^2 - \sin^2\phi} = 2\sqrt{\mu^2 - \sin^2\phi}$ nearly.

Hence we get

$$\sin\frac{\psi_v - \psi_r}{2} = \sin\frac{D_v - D_r}{2} = \frac{\mu \sin i\,(\mu_v - \mu_r)}{2\sqrt{\mu^2 - \sin^2\phi} \,.\, \cos\psi}$$

$$= \frac{(\mu_v - \mu_r)\sin i}{2\cos\phi' \,.\, \cos\psi},$$

which gives us the value of $D_v - D_r$, that is the angle subtended by rv, in Art. 109, at the point O.

It is clear that the linear distance rv will approximately $= \frac{f(\mu_v - \mu_r)\sin i}{\cos\phi' . \cos\psi}$, if f be the focal length of the lens O.

112. When a pure spectrum is obtained from the sun's light, it is found that it is not a continuous band of light, but that it has a very large number of interruptions, or places where the light does not exist. If the spectrum be formed from a slit, these interruptions appear as fine lines parallel to the edge of the prism, and consequently perpendicular to the length of the spectrum.

The number of these lines that can be seen increases with the number of the prisms, the purity of the material of which they are composed, and the fineness of the slit through which light is admitted. A very large number of them have been observed and their relative positions very accurately determined by measurement.

It is found that the same lines always occur in the same order, whatever may be the size or nature of the prism, and they are thus known as *the fixed lines of the solar spectrum.*

Some of the more important and well marked of these lines are taken as points of reference for the spectrum and denoted by the letters A, B, C, D, E, F, ..., and the position of any ray is determined by its position relative to these fixed lines.

For representations of the solar spectrum the reader is referred to those in Roscoe's Spectrum Analysis, and to other similar treatises.

113. It is found that the solar light reflected by the planets or the moon gives the same spectrum as the direct solar light, while that from the fixed stars presents different fixed lines from the solar spectrum.

Thus it appears that the solar spectrum is something essentially belonging to the sun itself.

The light from any glowing hot body, solid or fluid, with only a single known and perhaps problematical ex-

ception in the earth Erbia, is found to give a spectrum without any interruptions or fixed lines.

On the other hand, when by any means a body in a sufficiently attenuated gaseous state is rendered luminous, as by enclosing it in a glass tube, and passing the spark of an induction coil through it, the spectrum formed is very different, and consists of a number of isolated bright lines.

It is farther found that, when light from a highly heated incandescent body is made to pass through any gas of lower temperature than the source before falling on the spectroscope, the spectrum formed is no longer continuous, but has interruptions at precisely those points where bright lines would have been formed by the glowing gas itself if rendered sufficiently luminous.

We thus have a physical explanation of the fixed lines in the solar spectrum.

The body of the sun is supposed to be in a very hot semifluid or fluid state. This body would of itself give a continuous spectrum. It is supposed that this central body is surrounded by a cooler and therefore absorbent gaseous atmosphere, which contains portions of the elements existing in the sun's body which have been vapourised by the great heat.

The passage of the sun's light through this atmosphere produces the fixed lines in the spectrum.

By comparing the fixed lines in the sun's spectrum with the bright lines given by the vapours of various terrestrial substances, it has been inferred that several of these, as hydrogen, iron, sodium, &c., exist in a state of vapour in the sun's atmosphere, and that therefore probably those of them which can there assume a solid state exist in a solid or fluid state in the body of the sun.

Similar inferences have been made about the constitution of various orders and classes of the fixed stars.

The spectra given by some of the nebulæ consist of a small number of isolated bright lines. It is thence inferred that these nebulæ are really masses of luminous gaseous matter, in which nitrogen and hydrogen are the chemical elements most frequently observed.

114. It is clear from the last few Articles that in all cases where refraction takes place we shall have corresponding to each incident ray, not one, but an infinite number of refracted rays.

For instance, the formula,

$$\frac{1}{v}-\frac{1}{u}=(\mu-1)\left(\frac{1}{r}-\frac{1}{s}\right),$$

which gives the position of the focus of a pencil after direct refraction through a thin lens, will give a different value of v for each different value of μ. Hence corresponding to one original point of light there will be a series of points as images of it, of different colours, arranged along the axis of the lens.

The object-glass of an Astronomical Telescope will thus form a series of images of the original object, one behind the other and of different colours.

The question arises, is it possible to find a remedy for this and other effects which may arise from the fact of light being decomposed into its elements by refraction? Newton supposed that there was no remedy, and was hence led to turn his attention to the construction of reflecting telescopes. Later discoveries have showed that a remedy can be partially, if not perfectly, applied.

115. If a pencil of white light be passed through a prism, it is deviated and also separated into a number of different pencils of different colours. If a second prism of the same material and size be interposed so as to deviate the pencil equally in the opposite direction, it is found that the dispersion is also corrected, and the pencils of different colours all proceed in the same direction as the original pencil; and in fact reproduce white light, without any deviation from the original direction.

If the second prism be of a different material, and be of such a refracting angle as to produce the same deviation of the mean ray, as the former prism, but in an opposite direction, it is found that the dispersion is not completely corrected.

Thus in prisms of different materials, which give the same deviation, the dispersion is not always the same.

On the other hand, since in prisms of the same material the dispersion is found to increase with the deviation, it follows that it will be possible to take two prisms of different materials which shall give the same dispersion, with different deviations.

If then a pair of such prisms be placed so as to deviate a pencil successively in opposite directions, the dispersions will be equal and opposite, that is, on the whole there will be no dispersion, while the deviations will not be equal, or, there will be deviation without dispersion.

The above remarks are strictly true only if we consider two particular rays of the spectrum.

It is found, for instance, that if A, B, C represent three particular rays of the spectrum, the ratio of the dispersion of A and B to that of A and C is different for different media. Thus if with two prisms of different media we give A and C equal dispersions in opposite directions, the dispersions of A and B in the two prisms will not usually be equal and opposite.

Achromatic combinations will thus in general only exactly unite two rays, and will be only imperfectly achromatic.

The fact on which this imperfection depends is called *the irrationality of dispersion.*

116. Before proceeding to apply these principles to the investigation of Achromatic combinations of lenses we must define the *dispersive power* of a medium.

The dispersive power of a medium may be defined generally as the ratio which the difference of the deviations of any two rays at opposite ends of the spectrum bears to the deviation produced in a ray somewhere in the middle of the spectrum, which we may call the mean ray, when a ray of white light passes through a prism formed out of this medium.

This ratio varies with the angle of the prism. For the sake of precision we will therefore define the dispersive power of a medium as the limiting value of the above ratio

when the angle of the prism is indefinitely diminished. This will not differ much from its value for moderate angles of the prism.

Let μ be the refractive index of the medium for the mean ray, μ_v, μ_r its refractive indices for two rays at the violet and red ends of the spectrum respectively. Let D, D_v, D_r be the deviations of these rays.

Then the dispersive power of the medium is the limit of the ratio $\dfrac{D_v - D_r}{D}$ when the angle of the prism is indefinitely diminished.

But since we are finally to assume the angle of the prism indefinitely small, we may use the last formula in Art. 60, and we have

$$D = (\mu - 1)\, i,$$
$$D_v = (\mu_v - 1)\, i,$$
$$D_r = (\mu_r - 1)\, i.$$

Hence the dispersive power

$$= \text{limit of } \frac{D_v - D_r}{D} = \frac{\mu_v - \mu_r}{\mu - 1}.$$

This fraction is usually denoted by the symbol ϖ.

117. If two thin lenses be placed in contact with a common axis, and a pencil of light be directly refracted through the combination, the distances u and v of the point of light and its image respectively from the common centre of the lenses are connected by the formula

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f_1} + \frac{1}{f_2} \text{ (Art. 75)},$$

f_1, f_2 being the focal lengths of the lenses.

Now as above explained (Art. 114) f_1 and f_2 will differ for the blue and red rays, but if we can arrange the focal lengths so that $\dfrac{1}{f_1} + \dfrac{1}{f_2}$ shall have the same value for the two rays we select, the value of v will be the same for these

rays, and the corresponding images of the original point of light will coincide.

The value of $\frac{1}{f_1}$ for the red rays will be

$$(\mu_r - 1)\left(\frac{1}{r} - \frac{1}{s}\right),$$

and for the ray at the other end of the spectrum it will be

$$(\mu_v - 1)\left(\frac{1}{r} - \frac{1}{s}\right).$$

The difference between these is

$$(\mu_v - \mu_r)\left(\frac{1}{r} - \frac{1}{s}\right)$$

$$= \frac{\mu_v - \mu_r}{\mu - 1} \times (\mu - 1)\left(\frac{1}{r} - \frac{1}{s}\right) = \frac{\varpi_1}{f_1},$$

if ϖ_1 be the dispersive power of the medium of which this lens is composed.

Similarly, if ϖ_2 be the dispersive power of the second medium, the difference of the values of $\frac{1}{f_2}$ for the red and blue rays, will be $\frac{\varpi_2}{f_2}$.

Hence in order that the two images may coincide, we must have $\frac{\varpi_1}{f_1} + \frac{\varpi_2}{f_2} = 0$.

This, therefore, is the condition that two lenses placed in contact with a common axis may form an Achromatic combination. It is clear that f_1 and f_2 must have opposite signs; one lens must therefore be convex and the other concave.

The object-glass of an achromatic Astronomical Telescope is formed of two lenses, one concave and the other convex, placed in contact, the focal length of the convex being the shorter, so that the combination may be convex on the whole, and by the above formula the concave lens must have the greater dispersive power.

In consequence of the irrationality of dispersion such a combination will not unite all the images, but it can be made to unite any two of the most important complementary colours, and will in so doing bring the others nearer to each other.

118. The refraction through the eye-piece of an astronomical telescope being excentrical, the axes of the pencils undergo deviation, and this deviation will be different for the rays of different colours of which the pencil is composed.

To form an achromatic eye-piece it is necessary to arrange two lenses so that the deviations of two rays belonging to the two ends of the spectrum in passing through the combination shall be equal.

It will nearly ensure this if the value of the focal length of the equivalent lens investigated in Art. 76 is the same for the two rays. The investigation in that Article, it may be noticed, depends on the deviation of an excentrical ray incident parallel to the axis.

In practice two convex lenses are always employed, and we will therefore take the second formula in that article

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{a}{f_1 f_2},$$

f_1, f_2 being the numerical values of the focal lengths of the lenses.

As in the last article the differences in the values of $\frac{1}{f_1}$ and $\frac{1}{f_2}$ for rays at the two ends of the spectrum are $\frac{\varpi_1}{f_1}$ and $\frac{\varpi_2}{f_2}$ respectively.

Hence the whole alteration in the value of $\frac{1}{F}$ is

$$\frac{\varpi_1}{f_1} + \frac{\varpi_2}{f_2} - a\frac{(1+\varpi_1)(1+\varpi_2)-1}{f_1 f_2},$$

and this must vanish, if the combination is achromatic.

If the lenses be of the same material $\varpi_2 = \varpi_1$, and the condition for achromatism becomes

$$\frac{1}{f_1} + \frac{1}{f_2} - \frac{a(2+\varpi)}{f_1 f_2} = 0,$$

or since ϖ is a small quantity, approximately we have

$$\frac{1}{f_1} + \frac{1}{f_2} - \frac{2a}{f_1 f_2} = 0,$$

$$\therefore f_1 + f_2 = 2a.$$

This condition is satisfied by Huyghens' eye-piece, since in that combination the focal lengths are f and $3f$ respectively, and a is $2f$.

Ramsden's eye-piece is not achromatic, as it does not satisfy the above condition.

The above investigation only applies strictly to an excentrical ray incident parallel to the axis, but it will approximately hold for all rays which are not very oblique. The condition of achromatism, in fact, varies with the obliquity.

119. It is usual to add a proposition on the condition of achromatism of a pencil refracted through two prisms.

If the angles of the prisms be small, the condition is very simple and easily found.

Let i, i' be the angles of the prisms,

μ, μ' their refractive indices for the mean ray,

D, D' the deviations of the mean ray in passing through them,

ϖ, ϖ' their dispersive powers,

$$\therefore D = (\mu - 1)\, i, \quad D' = (\mu' - 1)\, i'.$$

Hence the deviation of the red ray on the whole will equal

$$(\mu_r - 1)\, i + (\mu_r' - 1)\, i',$$

and that of the violet ray on the whole will be

$$(\mu_v - 1)\, i + (\mu_v' - 1)\, i'.$$

But if the combination be achromatic these deviations must be equal;

$$\therefore (\mu_v - \mu_r)\, i + (\mu_v' - \mu_r')\, i' = 0,$$

or $$\frac{\mu_v - \mu_r}{\mu - 1} \cdot (\mu - 1)\, i + \frac{\mu_v' - \mu_r'}{\mu' - 1} \cdot (\mu' - 1)\, i' = 0;$$

$$\therefore \varpi D + \varpi' D' = 0.$$

The deviations must obviously therefore be in opposite directions and inversely proportional to the dispersive powers of the media.

The formula of Art. 111 will enable the student to deduce a condition of achromatism when a ray of white light passes through two prisms of finite angles.

EXAMPLES. CHAPTER IX.

1. If the prism in Newton's experiment be first placed in the position of minimum deviation for red rays, and afterwards in the position of minimum deviation for violet rays, examine in which case the longer spectrum will be obtained.

2. Shew that if the prism employed to produce a pure spectrum be not in the position of minimum deviation, a pure spectrum can be still produced on a screen. Must the screen be placed at the primary or secondary focus of the pencils after refraction through the lens in Art. 109?

3. If the screen be placed in a plane perpendicular to the direction of the light before it passes through the prism in Newton's experiment, prove that for a given position of the edge of the prism the length of the spectrum will be proportional to

$$\frac{(\mu_v - \mu_r) \sin i}{\cos^2 D \cos (D + i - \phi) \cos \phi'},$$

μ_v, μ_r being the refractive indices for the extreme rays, D the mean deviation, i the angle of the prism, and ϕ, ϕ' the angles of incidence and refraction at the first surface.

4. The refractive index of a medium for the two rays at the red and violet ends of the spectrum being 1·63 and 1·66 respectively, calculate the dispersive power.

5. Calculate the dispersive power of a medium for which the refractive indices for the same two rays are 1·53 and 1·54 respectively, and find the ratio between the focal lengths of two lenses formed of the media in this and the last example, that the combination may form an achromatic object-glass for an astronomical telescope.

6. Prove that, if f be the focal length of a lens, ϖ its dispersive power, v the distance from the centre of the lens of the point to which a pencil of mean rays is made to converge, the distance between the foci of red and violet rays for the same incident pencil is approximately $\dfrac{\varpi v^2}{f}$.

7. The dispersive power of a medium is ·036. The focal length of a lens formed of it being 3 feet for mean rays, find the distance between the extreme images of the sun formed by the lens.

8. The refractive indices of one medium for three particular rays of the spectrum are 1·628, 1·642 and 1·660 respectively. Those of another medium for the same rays are 1·525, 1·533 and 1·541 respectively. Show that these values exhibit a difference of dispersive power and also the irrationality of dispersion.

9. Prove that if ϕ be the angle of incidence of a ray of white light on a prism of mean refractive index μ and dispersive power ϖ, and ϕ_1 be the angle of emergence of the mean ray from a second prism of mean refractive index μ_1 and dispersive power ϖ_1, the combination will be achromatic if

$$\frac{(\mu-1)\varpi}{\mu} \cdot \tan\phi = \frac{(\mu_1-1)\,\varpi_1}{\mu_1} \cdot \tan\phi_1 .$$

CHAPTER X.

MISCELLANEOUS THEOREMS.

120. IN the present Chapter we shall collect some miscellaneous Propositions and experimental facts which do not exactly belong to the general train of investigation, but which are usually included in the subject of Geometrical Optics.

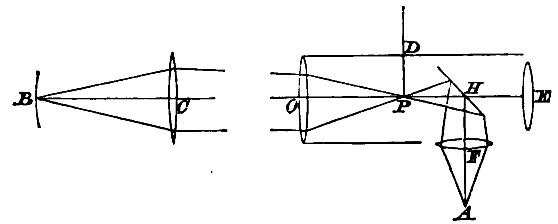
121. It is a well-known fact that light requires time for its propagation.

This fact is proved by two astronomical phenomena. It was found, by a comparison of different observations, that the eclipses of Jupiter's Satellites, which are phenomena of great importance in determining the longitude at sea, appeared to happen later or earlier than their calculated times, according as Jupiter was farther from, or nearer to, the earth. By comparing the times of happening of such eclipses when Jupiter was nearest to the earth, with their times when Jupiter was at his greatest distance from the earth, the time taken by light to travel a known distance, the diameter of the earth's orbit, was discovered, and hence the velocity of light was known.

A nearly equal value of the velocity of light was determined by the phenomenon of aberration; a small displacement in the position of a star which was discovered by Bradley to arise from the composition of the velocity of light with that of the earth. For a full account of this phenomenon the reader is referred to any treatise on Astronomy.

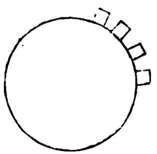
122. The velocity of light has also been determined by direct experiment in two ways, one of which we will describe. In this, which is known as Fizeau's method, from the name of its inventor, the light is made to travel from one station to another at a distance of two or three miles from the first and back again; and the time in which this distance is traversed is determined in the following manner.

Fig. (1).



O is the object-glass of a telescope, *E* its eye-piece. The telescope is placed at the one station, and is pointed to the other station, at which is placed a convex lens whose

Fig. (2).



centre is *C* and whose axis nearly coincides with that of the first telescope. At the focus of this lens behind it is placed a mirror *B* whose axis coincides with that of the lens *C*, and which is best made as a portion of a sphere whose centre is *C*.

Near the eye-piece of the telescope a small lens *F* is placed in the side of the tube of the telescope. Outside and in front of this lens, as at *A*, is placed a candle or

luminous point of some description. The rays from A fall on the lens F and are refracted by it so that they would converge to a point within the tube of the telescope. A piece of plane glass is placed to intercept these rays before they converge, and being placed at an inclination of 45^0 to the axis of the telescope, reflects the pencil so that its axis after reflection is parallel to that of the telescope. The rays will thus converge to some point, as P, and then diverge and fall upon the object-glass O. The positions of A, F, and H are so chosen that the point P is at a distance from O equal to the focal length of the object-glass. The pencil from P will thus emerge as a pencil of parallel rays, whose direction is parallel to PO. The light will proceed to the second station, fall upon the lens C, and be refracted to converge to a point on the mirror B. By this mirror the rays will be reflected back on to the lens C and again emerge as parallel rays in their original direction. They will come back to the lens O and be made to converge to the point P, whence again diverging, some of them will pass through the plane glass H and fall on the eye-piece E, thus giving to the eye vision of a bright point at P.

123. A wheel represented in Fig. (2) having a large number of equal teeth, the space between any two consecutive teeth being just equal to the width of one tooth, and which can rotate about an axis through its centre perpendicular to its plane, is placed with its plane at right angles to the axis of the telescope. It is so placed that as it rotates its teeth shall just pass through the point P and consequently, as the wheel is turned round, the light which comes into the eye will be stopped whenever a tooth is at P, and will pass when an opening is at P.

If the wheel be at rest with an opening at P, the light will pass out and in just as before; but if we can make the wheel turn with such a velocity that the light which went out through an opening shall just find a tooth in its way when it comes back, the light will be altogether prevented from reaching the eye.

It is clear that if there be n teeth and n openings, the wheel must be turned through an angle $\frac{\pi}{n}$ while the

light travels from one station to the other and back, in order that the light which went out at each point of any opening may be stopped by the corresponding point of the next tooth.

If therefore we can measure the rate of rotation of the wheel when it is moving just so fast as to stop all light in returning, we can determine the velocity of light.

This was done by M. Fizeau, and the result agreed very nearly with that obtained from Astronomical observations, but gave a result very slightly smaller. The velocity determined by this experiment is in fact the velocity of light in air, while that determined from Astronomical phenomena is nearly its velocity in a vacuum.

124. It is clear that if the velocity of the wheel either slightly fall short of, or slightly exceed, this particular velocity, some portion of the light will get through.

Thus, if we begin by making the wheel revolve slowly, only that portion of light which passes out through the last part of an opening will be stopped in returning; but, inasmuch as half the light will be stopped in passing out by the teeth, either the appearance will be that of an intermittent light if the velocity be very slow, or will be a continuous light of less than half the brightness which it had when the wheel was stationary, if the rotation be so rapid that a continuous impression is produced on the retina.

As the wheel rotates more rapidly, more of the light which goes out at any opening will be stopped by the next tooth, and the image will gradually grow fainter, until we reach the exact velocity which causes all light to be stopped.

If the wheel be made to revolve still faster, some of the light which went out at the end of one opening will come back through the beginning of the next opening, and a faint image will reappear, which will continually increase in brightness as the velocity of the wheel is increased, until, when this velocity is just double of that

which produced the total eclipse, all the light which passed out through one opening will return through the next.

An image of half the brightness of the original point of light will thus be seen.

Making the wheel revolve still faster, some of the light will begin to be stopped by the second tooth, and the brightness of the image will decrease, until, with a velocity three times that which gives the first eclipse, all the light which passes out at any opening will be stopped in returning by the second tooth, and there will be a second total eclipse.

By proceeding in this way, a series of maxima of brightness, alternating with eclipses, will occur, and by measuring the velocities of rotation of the wheel which produce them, we can obtain a series of independent determinations of the velocity of light.

125. The other method of determining the velocity of light by experiment depends on the use of a revolving mirror. Its special value consists in determining the difference between the velocities of light in air, and in passing through a dense medium, as water, respectively.

A description of this method, known as Foucault's method, can be found in Billet's *Traité d'Optique Physique*, § 36, or in Parkinson's *Optics*, Art. 47*. We shall not describe it here.

126. When light from any source falls on any surface, some portion of the light is scattered and makes the surface visible. The proportion of the quantity of light scattered to the whole quantity that is incident, depends very much on the nature of the surface. Thus, when light falls on a smooth piece of glass, scarcely any of it is scattered, while, when the same light falls on a piece of white paper, a very large part of it is scattered, and the paper appears brightly luminous.

This scattered portion may, however, when the nature of the surface remains unchanged, be assumed to be proportional to the whole quantity of light which is incident,

and there are one or two propositions in relation to it which are of some interest.

127. We must first give the following definitions:

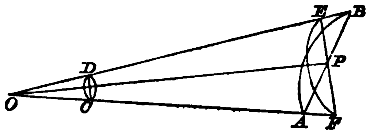
The illumination at any point of a surface, uniformly illuminated from any source of light, is measured by the quantity of light scattered by a unit of area of the surface.

The illumination at any point of a surface, not uniformly illuminated, is measured by the quantity of light which would be scattered by a unit of area of the surface, supposed illuminated uniformly with the same intensity as the point considered.

Thus, if $I\kappa$ be the whole quantity of light scattered by a portion of the surface, whose area κ is so small that it may be supposed uniformly illuminated, I will clearly be the illumination of a unit of area equally illuminated with this small element, and will thus be the measure of the illumination at any point of the element of area.

128. If a pencil of light proceed from any point, and we imagine sections of this pencil made at various distances and inclinations, the quantity of light which falls on planes placed so as to coincide with these sections will be the same; and thus the illumination at any point of any one of them will evidently vary inversely as the area of the section, always supposing the pencil so small that the illumination may be supposed uniform over each section.

Let us now suppose O to be the vertex of such a pencil, which we will assume to be in the form of a right cone whose axis is OP.



Let I measure the illumination at any point of a section CD of this cone, made by a plane at right angles to OP, at a unit of distance from O.

Let I' represent the illumination at any point of a section AB of this cone, made by a plane which cuts the axis of the cone at a distance OP from O, and is inclined to OP at an angle $\frac{\pi}{2}-\theta$, so that θ is the angle of incidence of the light on this plane, that is, the angle between OP and the normal to this plane.

Then, the quantity of light falling on these two sections being the same, we have

$$I : I' :: \text{area of section } AB : \text{area of section } CD.$$

But if we draw through P a section EF at right angles to OP, this section will be to the section CD in the ratio of the squares of their radii, that is, in the ratio of the squares of their distances from O;

$$\therefore \text{ area of section } EF = \text{area of section } CD \times OP^2.$$

Again, since we suppose the solid angle of the cone to be very small, we may suppose the section EF to be the orthogonal projection of AB on the plane EF.

Hence, by the theory of projection,

$$\text{area of } EF = \text{area of } AB \times \cos\theta;$$

$$\therefore \text{ area of section } CD = \text{area of } AB \times \frac{\cos\theta}{OP^2}.$$

But $$\frac{I'}{I} = \frac{\text{area of section } CD}{\text{area of section } AB}.$$

Hence $$I' = I \cdot \frac{\cos\theta}{OP^2}.$$

Hence since, if the source of light and the material of the surface on which the light falls be given, the value of I is a definite quantity, we may say

$$I' \propto \frac{\cos\theta}{OP^2};$$

that is, the illumination produced at any point of a surface of given material by a given source of light, varies

directly as the cosine of the angle of incidence, and inversely as the square of the distance of the point of the surface from the source of light.

129. By means of this principle, instruments have been devised for comparing the intensities of different illuminating sources.

It is clear that for different sources of light the quantity I of the last article will be proportional to the intensity of the source, and may therefore be taken as a measure of the intensity in any particular case.

If then we dispose two screens of the same material in such a manner that the illumination produced at definite points of them by the two sources of light shall be equal, we shall only have to measure the values of OP and $\cos\theta$ for the two sources of light, and, the values of I' being the same in the two cases, the ratio of the two values of I will be obtained.

This method is practicable because, although the eye is not able to judge of the ratio of the intensities of two illuminations which differ from each other, it can judge tolerably accurately of the equality of two illuminations.

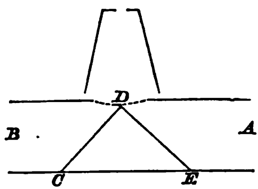
130. One very simple method is to allow a screen to be lighted by both sources of light, and to place a stick between this screen and the two sources. Each light will cast a shadow of the stick on the screen, and the lights must be moved until these shadows appear of equal darkness. The screen being lighted by both lights and each shadow only lighted by one, it is clear that when this is the case the illuminations produced on the screen by the lights will be equal; and if the lights be so placed that the angle of incidence is the same, the illuminating powers of the lights will be directly as the squares of their distances from the screen.

131. Another contrivance for effecting the same thing consists of a screen of paper, the greater portion of which is rendered translucent by being soaked with oil, while a small circular portion in its centre is left opaque. The two

lights whose intensities are to be compared are placed on opposite sides of this screen. If the lights be placed at equal distances from the screen, the opaque part of the screen will appear brighter than the translucent part on that side on which is the stronger light, while on the other side, the reverse will be the case. A reflector is placed on each side of the screen, inclined at an angle of 45^0 to it, so that an observer standing opposite to the edge of the screen can see the reflections of the two sides simultaneously. If the lights are adjusted so that the illuminations of the two sides appear equal, the intensities of the lights are directly proportional to the squares of their respective distances from the screen.

132. A possible disadvantage attending the arrangement described in the last Article is, that the reflections of the two sides of the screen are seen one by the right eye and the other by the left eye of the observer, and the two eyes of the same person are seldom of exactly the same power. This defect is obviated in Ritchie's Photometer.

This consists of an oblong box open at each end; about the middle of this box are placed two reflectors *CD*, *ED* inclined at 45^0 to its length, and cut from the same piece



of glass, to ensure equality of reflecting power; just above the line of intersection of these mirrors and parallel to this line is a slit in the top of the box, covered with a piece of parchment or paper.

The lights to be compared are placed, one opposite to each end of the box, so that the light from them falls on the mirrors *CD* and *DE*, and is reflected so as to illuminate the parchment which covers the slit. An eye looking

down on this parchment, through a tube blackened internally so as to prevent extraneous light from interfering, will see the two parts of the parchment illuminated by the two sources of light respectively. The lights must be moved till these two parts appear equally bright; the intensities of the lights are directly proportional to the squares of their distances from the mirrors when this is the case.

133. The calculation of the illumination produced at a given point of a surface by a finite illuminating surface involves a principle which may be thus stated:

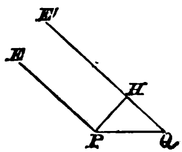
Any self-luminous surface of uniform brightness appears equally bright, whatever may be its distance from the eye and at whatever angle it may be inclined to the line of sight.

For instance, in looking at a mass of uniformly heated glowing iron in a furnace, the eye is unable to detect any variation in the apparent brightness of the different portions of iron due to their different distances or different inclinations to the line of sight.

Another illustration is afforded by the the fact that the different portions of the sun's disc appear equally bright, although they are at different distances and inclinations to the line of sight.

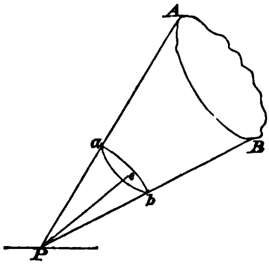
Thus we may say generally that any portion of a luminous surface sends as much light to the eye as any other portion of the same surface, at whatever distance, or however placed, which subtends the same solid angle at the eye.

As a particular case of this proposition, any element of the surface will send out obliquely an amount of light



which is to that which it emits directly, in the ratio of $\sin\theta$ to unity, θ being the angle which the direction of emission makes with the surface. For if PQ be any element of the surface, and PE the oblique direction of emission, the amount of light emitted in that direction will only be the same as would be emitted directly by a portion of the surface PH, where QH is drawn parallel to, and PH perpendicular to, PE. But $PH = PQ\sin\theta$, whence the required result follows.

134. Let AB be any uniform illuminating surface, and let P be any point in a plane at which the illumination is required. With P for vertex and with the boundary of AB for base, describe a cone. Also with centre P and any radius Pa describe a sphere which will cut this cone



in some curve ab. If the portion of the surface of the sphere contained within this curve were supposed to be of illuminating power equal to that of AB, by the last Article the illumination produced by it at P would be the same as that produced by AB.

Let e be any small element of this spherical surface, and let θ be the angle which eP makes with the plane at P. Then, if C be the intrinsic illuminating power of AB, the illumination produced by e at P will, by Art. 128,

$$= C.e.\frac{\sin\theta}{(Pe)^2} = \frac{C.e\sin\theta}{Pa^2}.$$

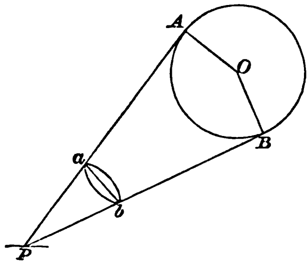
But $e \sin\theta$ is the projection of the small area e on the plane in which P lies, since $\frac{\pi}{2}-\theta$ is the angle between that plane and the element e. Hence the whole illumination produced by AB at P

$$=\frac{C\times \text{projection of surface included within the curve } ab \text{ on plane at } P}{Pa^2},$$

a formula which can often be made of use in calculation.

135. A useful and important example of the last Article is found in the investigation of the illumination produced at any point of a surface by a uniformly luminous sphere.

Let O be the centre of the luminous sphere, a its radius. Let P be the point in the surface at which the illumination is required to be found, and let θ be the angle between OP



and the tangent-plane to the surface at P, that is, the plane with which a small element of the surface at P may be supposed to coincide (Art. 9). Let $OP=c$.

Let lines be drawn from P touching the sphere. It is clear that all these lines will lie on a right circular cone whose vertex is P and whose semivertical angle is the angle whose sine is $\frac{a}{c}$. With centre P, and any radius Pa, describe a sphere, which will cut the cone in a circle ab. The projection of the portion of the spherical surface in-

cluded by this circle on the plane at P is the same as the projection on the same plane of the area of the circle itself, and therefore equals

$$\pi . Pa^2 . \sin^2 OPA \sin\theta = \pi . Pa^2 . \frac{a^2}{c^2} \sin\theta.$$

Hence the illumination at P produced by the sphere

$$= \frac{C}{Pa^2} \times \frac{\pi . Pa^2 . a^2}{c^2} \sin\theta$$

$$= \frac{C\pi a^2}{c^2} \sin\theta.$$

This expression gives the illumination *at a point* P. The illumination on the element of surface containing P is obtained by multiplying this expression by the area of the element.

EXAMPLES. CHAPTER X.

1. If the wheel in Fizeau's experiment have 720 teeth and make $21\frac{1}{4}$ turns in a second, when the first eclipse takes place, find the velocity of light; the distance between the stations being three miles.

2. The light from two sources is allowed to fall on the same screen. One light is at a distance a and the light falls directly from it on the screen. From the other, which is at a distance $3a$, the light falls at an obliquity of 60^0. The illuminations of the screen from the two sources being equal, compare the intrinsic brightness of the two lights.

3. A luminous point is placed at the focus S of an ellipse. Two focal chords PSp and QSq are drawn. Shew that the sum of the illuminations of the arcs PQ and pq is the same as long as the angle PSQ is the same.

Hence find the whole illumination of the perimeter of the ellipse.

4. A very narrow band of uniform breadth κ is bent into the form of an elliptical hoop. A luminous sphere, of radius a and intrinsic brightness I, is placed with its centre at the focus of the ellipse. Shew that the whole illumination of the hoop is equal to $\frac{4\pi^2 a^2 \kappa I}{L}$, where L is the latus rectum of the ellipse.

5. Find the position of a bright point which equally illuminates the three sides of a given triangle.

6. A small plane area is placed at right angles to the axis of a paraboloid of revolution whose convex surface is uniformly luminous. Prove that the illumination produced at the point of the plane where it meets the axis varies inversely as the distance of this point from the focus of the paraboloid.

7. A small plane area is placed parallel to a plane lamina of intrinsic brightness I, of breadth $2a$, and of infinite length, at a distance c from the centre of the lamina in a line perpendicular to the lamina. Prove that the illumination at the centre of the plane area is $\frac{\pi a I}{\sqrt{a^2+c^2}}$.

8. Shew how to calculate the illumination produced by a window on a point of the floor directly in front of the centre of the window: the window being supposed to reach to the level of the floor.

9. Two spheres are luminous, and a small plane area is placed on a line joining their centres, its plane being perpendicular to this line. Find where it must be placed in order that its two surfaces may be equally bright.

10. Three equally bright points are placed at the angular points of an equilateral triangle. If a plane area be placed at the centre of the triangle in any manner, shew that it will be equally bright on both sides.

11. A triangular prism, whose nine edges are all equal, is placed with one of its rectangular faces on a horizontal table, and illuminated by a sky of uniform brightness; shew that the total illuminations of the inclined and vertical faces are in the ratio of $2\sqrt{3}$ to 1.

12. A luminous point is placed on the axis of a truncated conical shell; prove that the whole illumination of the shell varies as

$$\frac{c_2}{(c_2^2+a_2^2)^{\frac{1}{2}}} \pm \frac{c_1}{(c_1^2+a_1^2)^{\frac{1}{2}}}$$

where a_1, a_2 are the radii of the circular ends of the shell, and c_1, c_2 the distances of the luminous point from their planes.

13. The sides of a triangle are the bases of three infinite rectangles of the same brightness, whose planes are perpendicular to the plane of the triangle: shew that all points within the triangle are equally illuminated. Find the position of a point in the plane of the triangle, such that the illuminations at that point received from the three rectangles may be equal.

CHAPTER XI.

THE RAINBOW.

136. IN this Chapter we propose to give a brief explanation of the formation of a Rainbow, as far as it can be done by the principles of Geometrical Optics.

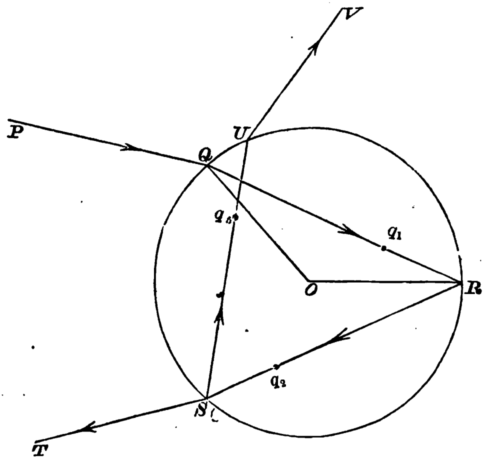
137. If a pencil of parallel rays falls on a refracting sphere, some portion of the light will be reflected externally, some portion will be refracted into the sphere. Of this latter part, when it is incident internally at the surface of the sphere, some portion will emerge, and another portion will be reflected internally, and be again incident on the internal surface of the sphere. At this second incidence the same division will again take place, and so on, at each successive internal incidence..

The primary and secondary rainbows are produced by portions of sunlight which, having been incident on raindrops, emerge after one or two internal reflections respectively.

We have therefore to consider mainly the circumstances attending the refraction and reflection of these portions in the case of light incident on a sphere of water.

138. Let PQ be the axis of a pencil of parallel rays incident on a refracting sphere at Q, refracted at Q along QR, reflected internally at R, and again incident at S. Let ST be the direction of that part which emerges at S,

and SU the direction of the reflected part, which is incident internally again at U, where some part of it emerges along UV.



We shall first examine the positions of the primary foci of the pencils emerging at S and U respectively.

Let ϕ be the angle of incidence at Q, ϕ' the angle of refraction. It is plain that ϕ' will also be the angle of incidence at R, S and U, and that ϕ will be the angle of emergence at S or U.

Let μ be the index of refraction, r the radius of the sphere. Let q_1, q_2, q_3 be the primary foci after refraction at Q, reflection at R, and emergence at S respectively. Let q_5, q_6 be the primary foci after reflection at S and emergence at U respectively. Then, by Art. 46, since the incident pencil consists of parallel rays,

$$Qq_1 = \frac{\mu r \cos^2 \phi'}{\mu \cos \phi' - \cos \phi}.$$

But $QR = 2r\cos\phi'$; $\therefore\ Rq_1 = 2r\cos\phi' - \dfrac{\mu r\cos^2\phi'}{\mu\cos\phi' - \cos\phi}$

$$= r\cos\phi' \cdot \frac{\mu\cos\phi' - 2\cos\phi}{\mu\cos\phi' - \cos\phi}.$$

And by Art. 45,

$$\frac{1}{Rq_2} + \frac{1}{Rq_1} = \frac{2}{r\cos\phi'},$$

whence $$Rq_2 = r\cos\phi' \cdot \frac{\mu\cos\phi' - 2\cos\phi}{\mu\cos\phi' - 3\cos\phi};$$

$$\therefore\ Sq_2 = r\cos\phi' \cdot \frac{\mu\cos\phi' - 4\cos\phi}{\mu\cos\phi' - 3\cos\phi}.$$

But by Art. 46,

$$\frac{\frac{1}{\mu}\cos^2\phi}{Sq_3} - \frac{\cos^2\phi'}{Sq^2} = \frac{\frac{1}{\mu}\cos\phi - \cos\phi'}{r},$$

whence $$Sq_3 = \frac{r\cos\phi\,(\mu\cos\phi' - 4\cos\phi)}{2\,(\mu\cos\phi' - 2\cos\phi)} \quad \text{............(1)}.$$

By proceeding in this way it will be easy to obtain a second result

$$Uq_5 = \frac{r\cos\phi\,(\mu\cos\phi' - 6\cos\phi)}{2\,(\mu\cos\phi' - 3\cos\phi)} \quad \text{............(2)}.$$

We may notice incidentally that Sq_3 and Uq_5 respectively become infinite, that is, the rays in the primary plane emerge as a pencil of parallel rays after one or two internal reflections, when

$$2\cos\phi = \mu\cos\phi',$$
$$3\cos\phi = \mu\cos\phi',$$

respectively.

139. We shall now restrict ourselves to the consideration of the portion of light which emerges after one internal reflection.

The deviation of the axis of the pencil at Q is clearly $\phi - \phi'$: at R its deviation is $2\pi - 2\phi'$; and at S it under-

goes a farther deviation in the same direction of $\phi - \phi'$. Its deviation on the whole is therefore

$$\begin{aligned} & 2(\phi-\phi')+2\pi-2\phi' \\ &= 2\pi - 2(2\phi'-\phi) \\ &= 2\pi - 2\{\phi' - (\phi-\phi')\}. \end{aligned}$$

Now we know by Articles 56 and 62, that $\phi - \phi'$ increases as ϕ' increases, but that for a given increment given to ϕ', the increment of $\phi - \phi'$ is larger, the larger the value of ϕ'.

Hence if we take a series of pencils whose axes are incident on the sphere at different angles, beginning with direct incidence and gradually increasing the obliquity, $\phi - \phi'$ and ϕ' will both increase, but at first the increment of $\phi - \phi'$ will be less than that of ϕ', while finally the increment of $\phi - \phi'$ may be greater than that of ϕ'. Hence in this case there must be some value of ϕ' for which the increment of $\phi - \phi'$ is just equal to that of ϕ'.

When ϕ' is small we thus have on the whole $2\phi' - \phi$ increasing and therefore the deviation of the axis decreasing, while when ϕ' is large we have $2\phi' - \phi$ decreasing, and thus the whole deviation increasing.

There is therefore a value of ϕ' such that the deviation is a minimum, and this value is evidently given by the above condition that the alteration in $\phi - \phi'$ for a given small change of ϕ' is exactly equal to the alteration in the value of ϕ'.

This value of ϕ' we proceed to investigate.

140. Let ϕ' be changed to $\phi' + a$ where a is small, then by the above condition, the new value of $\phi - \phi'$ must exceed the old value by a. Hence if ϕ_1 be the new value of ϕ, we must have

$$\phi_1 - (\phi' + a) = (\phi - \phi') + a;$$

$$\therefore \phi_1 = \phi + 2a;$$

$$\therefore \sin(\phi + 2a) = \mu \sin(\phi' + a);$$

but $$\sin\phi = \mu \sin\phi';$$

$$\therefore \sin(\phi + 2a) - \sin\phi = \mu\{\sin(\phi' + a) - \sin\phi'\};$$

$$\therefore 2\cos(\phi + a) . \sin a = 2\mu \cos\left(\phi' + \frac{a}{2}\right) . \sin\frac{a}{2};$$

$$\therefore 2\cos(\phi + a) . \cos\frac{a}{2} = \mu\cos\left(\phi' + \frac{a}{2}\right);$$

or, since a is to be a very small increment, this gives us

$$2\cos\phi = \mu\cos\phi' \ldots(1).$$

If a real value of ϕ' can be determined from this equation, for that value the deviation will be a minimum. It will be noticed that (1) is also the condition that the oblique pencil may emerge after one internal reflection as a pencil of parallel rays (Art. 138). We have also

$$\sin\phi = \mu\sin\phi' \ldots(2).$$

Squaring and adding these two equations we obtain

$$4\cos^2\phi + \sin^2\phi = \mu^2;$$

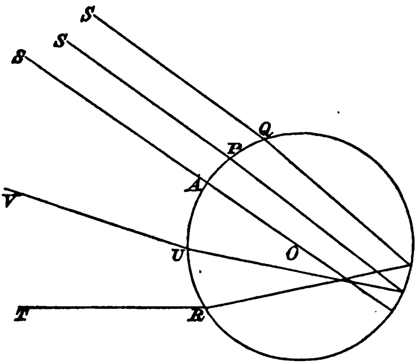
$$\therefore 3\cos^2\phi = \mu^2 - 1;$$

$$\therefore \cos\phi = \sqrt{\frac{\mu^2 - 1}{3}};$$

which gives the angle of incidence for a minimum deviation. In order that the value of ϕ may be real, $\mu^2 - 1$ must be less than 3, or μ must be less than 2. This is the case with water.

141. Let us now consider a beam of sunlight incident on the surface of a raindrop. Let O be the centre of the drop, SO the direction of incidence of the sun's light. Let SQ be that ray which is incident on the drop at the angle ϕ discovered in the last article, SP any other ray.

The ray which comes in the line SO is reflected back along OS, and its deviation is 180°. As the point of incidence passes from A towards Q, the deviation decreases and the ray which emerges after one internal reflection comes out in a direction continually more and more inclined to OS, until, when the ray is incident at Q, the direction of

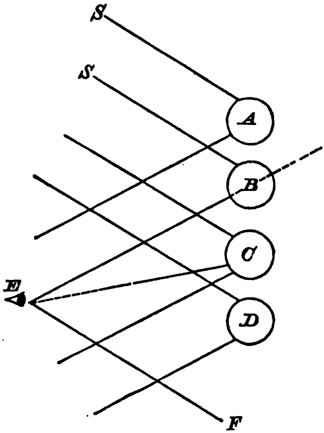


emergence attains its greatest angular distance from OS; when the point of incidence passes Q, the deviation again increases and the ray emerges at a less inclination to OS.

Also, considering the whole beam of sunlight which comes on to the upper half of the drop, as made up of a number of small pencils, the pencil whose axis passes with minimum deviation alone emerges as a pencil of parallel rays in the primary plane.

Thus if an eye be placed at a considerable distance from this drop, the only light which will produce any great impression on the eye will be that which emerges with minimum deviation, as all the other small pencils will have diverged considerably before they reach the eye, and only a very small portion of them will enter the eye.

142. Let us now suppose A, B, C, D to be a number of raindrops arranged in a vertical line, and let sunlight come upon them all. Let an eye be so placed that light, which is emergent from B with minimum deviation after one internal reflection, enters it. From A, which is above B, it is clear that no light will enter the eye, while from B strong light will enter the eye. From C some faint light will enter the eye, caused by divergent pencils whose deviation is slightly greater than the minimum, and a little sensible



illumination from drops a little lower still. Thus the eye will see a bright point in the direction of B, while there will appear to be no light above B, but light rapidly diminishing in intensity below B.

Through E draw a line EF parallel to the direction of the incident sunlight. Then it is clear that, if we draw any plane through EF inclined to the vertical, a similar appearance will be produced by the raindrops which at any instant lie in this plane.

Thus on the whole there will be seen a portion of a ring of light, whose apparent angular radius is the angle BEF, and whose centre is in the line EF, that is, exactly opposite to the direction of the sun; outside this ring is comparative darkness, while within it there is a certain amount of illumination, decreasing as we recede from the border.

143. We have hitherto taken no account of the different refrangibilities of the various rays of which sunlight is composed.

The angle ϕ, obtained in Art. 140, and consequently the minimum deviation, will however be different for different values of μ. The statements of the last Article will be true for each kind of light, but the size of the rings being different for different values of μ, there will not be a bow of white light but a series of bows, partly overlapping each other, corresponding to the different rays of the spectrum. The only important point to be investigated is the variation of the size of the ring for different values of the refractive index.

The angular radius of the bow in any case is easily seen from the figure in Art. 141 to be $2(2\phi'-\phi)$. In fact the bow will be largest when the minimum deviation is least.

$$\begin{aligned}\text{Now } \sin(2\phi'-\phi) &= \sin 2\phi' \cos\phi - \cos 2\phi' \sin\phi \\ &= 2\sin\phi'\cos\phi'\cos\phi - (\cos^2\phi' - \sin^2\phi') \,.\, \mu\sin\phi' \\ &= \mu\sin\phi'\cos^2\phi' - \mu\sin\phi'(\cos^2\phi' - \sin^2\phi'),\end{aligned}$$

(since when the deviation is a minimum $\mu\cos\phi' = 2\cos\phi$),

$$= \mu\sin^3\phi'$$

$$= \frac{\sin^3\phi}{\mu^2}.$$

$$\text{But } \cos\phi = \sqrt{\frac{\mu^2-1}{3}}; \quad \therefore \sin\phi = \sqrt{\frac{4-\mu^2}{3}}.$$

Hence if a be the angular radius of the bow corresponding to any value of μ, we have

$$\sin\frac{a}{2} = \frac{(4-\mu^2)^{\frac{3}{2}}}{3\mu^2\sqrt{3}}.$$

It is evident from this formula that the greater the value of μ, the less will be the value of a. Hence the red bow has the largest radius. Within the red bow will be a series of bows, of colours somewhat mixed but gradually varying from red to violet. Inside this coloured border will be a space of sky sensibly brighter than the average,

while beyond the red margin the sky will appear comparatively dark.

These phenomena are not affected by the fact that the raindrops are not stationary. If a shower of rain be falling and sunlight be incident on the drops, a series of drops in rapid succession will appear in the direction of B in Art. 142, and a continuous impression will be produced on the eye.

144. A somewhat similar investigation applies to the light which emerges after two internal reflections, and which produces the secondary rainbow. The deviation of the ray in this case is

$$4\pi - 2(3\phi' - \phi).$$

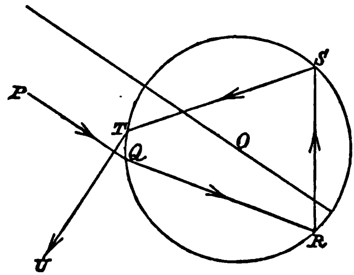
And by reasoning similar to that in Art. 139, it can be shewn that the deviation will have a minimum value when

$$3\cos\phi = \mu\cos\phi';$$

whence $$\cos\phi = \sqrt{\frac{\mu^2-1}{8}}.$$

For this value of ϕ, a small pencil will consist of parallel rays in the primary plane on emergence after two internal reflections (Art. 138).

In the accompanying figure PQ is the incident ray which passes with minimum deviation, and finally emerges along TU. The deviation being $4\pi - 2(3\phi' - \phi)$, it is clear



that the angular radius of the bow is less than this by 2π, or is equal to $2\pi - 2(3\phi' - \phi)$.

It will be found that if A be a drop from which light enters the eye with minimum deviation after two internal reflections, no light will enter the eye from drops below A, while faint light will reach the eye from drops above A. Thus the space within the secondary bow will be darker, while outside the bow will be light gradually fading away.

The minimum deviation can be shewn to increase with μ. Hence the radius of the bow, which increases with the minimum deviation, is greatest for violet light and least for red light.

The radius of the secondary bow is considerably larger than that of the primary bow.

145. Rainbows may theoretically be produced by light which has been internally reflected more than twice, but the intensity of the light diminishes so rapidly at each reflection that they cannot be seen practically with sunlight.

The theoretical investigation of such rainbows can be carried out in a similar way to that in which we have discussed the primary rainbow.

MISCELLANEOUS EXAMPLES.

1. If a pencil of parallel rays be incident obliquely on a refracting sphere, and emerge after one internal reflection with the least possible deviation, prove that the distance between the primary and secondary foci after the first refraction is $\dfrac{(4-\mu^2)r}{\mu\sqrt{3\mu^2-3}}$; r being the radius of the sphere.

2. Shew that if β be the angular radius of the secondary bow corresponding to any value of μ,

$$\sin\frac{\beta}{2}=\frac{\sqrt{(\mu^2-1)\,(9-\mu^2)^3}}{8\mu^3}.$$

3. If bubbles of air were rising in water, would a fish see a bow corresponding to a rainbow?

If drops of liquid of mean refractive index 2 were falling in the air, what would be the order of colours in the bow which would be formed?

4. A very short-sighted person, who is capable of seeing nothing distinctly beyond 3 inches, is able to see distinctly a small object distant $3\frac{1}{4}$ inches, through a pane of glass whose refractive index is $\frac{3}{2}$: find the thickness of the glass.

5. Four convex lenses, whose focal lengths are a, b, b, a respectively, are placed at intervals $a+b$, $2b\dfrac{a+b}{a-b}$, $a+b$, on the same axis: shew that a pencil of light, after refraction through all four lenses, diverges from the point from which it originally emanates.

6. A lens is moving with velocity p perpendicular to its axis, and an object at a distance a from the lens is moving with a velocity q across the axis in the opposite direction. Find the focal length of the lens, that to an eye on the other side of the lens the object may appear at rest.

7. A cylinder is made of a transparent surface whose refractive index is greater than $\sqrt{2}$; shew that when it is looked into by an eye situated anywhere on its axis produced, the whole of the inner curved surface will glisten brightly as compared with the inside of the opposite end.

8. Two rays proceed from the foci S, H of an ellipse, along the lines SP and HQ and are reflected at the ellipse; find the position of a plane mirror so placed that each ray after reflection at the mirror may return to the focus; and prove that the sum of the lengths of the paths of the rays is constant for all positions of P and Q.

9. A ray is refracted from vacuum through a series of plates, whose refractive indices are such that the ray suffers an equal amount of deviation a at each boundary. If μ_r be the absolute refractive index of the r^{th} medium, prove that $\mu_r \sec a$ is a harmonic mean between μ_{r-1} and μ_{r+1}.

10. Two parabolas have a common focus and axis, and their concavities are turned in the same direction. The inner surface of the outer parabola and the outer surface of the inner can reflect light. A ray of light starts from a point P on the outer towards the focus, and after $2n$ reflections strikes the inner again at Q. Shew that the distance traversed by the ray is greater than the difference of the distances of P and Q from the focus by n times the difference of the latera recta.

11. A transparent sphere is silvered at the back, prove that the distance between the images of a speck within it formed (1) by one direct refraction, (2) by one direct reflection and one direct refraction is $\dfrac{2\mu ac}{(a+c-\mu c)\,(\mu a+c-3c)}$, a being the radius of the sphere, and c the distance of the speck from the centre, measured towards the silvered side.

12. Q is a luminous point situated anywhere on the circumference of a reflecting circle, QP is any ray incident at P; PQ' is the chord of the circle in the direction of the reflected ray. If q be the point in which this reflected ray cuts the ray reflected from a point consecutive to P, prove that $Q'q=2Pq$.

13. A ray of light, traversing a homogeneous medium, is incident on a globular cavity within it: supposing the limit of

the magnitude of the deviation of the ray, produced by its passing through the cavity, to be θ, prove that the index of refraction of the medium is $\sec \frac{\theta}{2}$.

14. A reflecting polygon of an even number of sides can be inscribed in a circle: prove that if a ray of light proceeding from a point in any side returns to the same point after having been reflected at each side in succession, it will retrace its path.

15. An eye is placed close to the surface of a sphere of glass which is silvered at the back; the refractive index from air to glass being $\frac{3}{2}$, prove that the image which the eye sees of itself is $\frac{3}{5}$ of the natural size.

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